THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-fifth regular meeting of the Society was held in New York City on Saturday, October 25, 1913, extending through the usual morning and afternoon sessions. The following thirty-three members were present:

Professor R. C. Archibald, Dr. F. W. Beal, Professor E. W. Brown, Professor F. N. Cole, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Mr. G. H. Graves, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Mr. B. E. Mitchell, Mr. F. S. Nowlan, Professor W. F. Osgood, Dr. H. W. Reddick, Professor L. P. Siceloff, Dr. Clara E. Smith, Professor D. E. Smith, Mr. F. H. Smith, Professor F. B. Van Vleck, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair, being relieved at the afternoon session by Ex-President H. S. White. The Council announced the election of the following persons to membership in the Society: Mr. R. W. Burgess, Cornell University; Dr. Tomlinson Fort, University of Michigan; Dr. Cora B. Hennel, Indiana University; Mr. J. H. Kindle, University of Cincinnati; Professor Arthur Korn, Charlottenburg, Germany; Mr. M. A. Linton, Provident Life and Trust Company, Philadelphia, Pa.; Mr. John McDonnell, Canadian Geodetic Survey; Mr. J. Q. McNatt, Colorado Fuel and Iron Company, Florence, Colo.; Mr. T. E. Mason, Indiana University; Mr. B. E. Mitchell, Columbia University; Mr. George Paaswell, New York City; Mr. D. M. Smith, Georgia School of Technology; Professor Panaiotis Zervos, University of Athens. Twelve applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted.

In response to an invitation from Brown University to

participate in the celebration of its one hundred and fiftieth anniversary, it was decided to hold the summer meeting of the Society in 1914 at that university.

A committee consisting of the Secretary and Professors Dickson and Osgood was appointed to take charge of the publication of the Madison Colloquium Lectures. The volume, which should appear in a few months, will be designated as Volume IV of the series of published Colloquia, its predecessors being the Boston Colloquium Lectures, the New Haven Mathematical Colloquium, and the Princeton Colloquium Lectures, published in 1905, 1910, and 1913, respectively.

In view of the equal importance of the meetings of the Chicago Section with those held in New York and technically described as meetings of the Society, the Council, in the exercise of the powers conferred upon it by By-Law III, has now designated the meetings of the Chicago Section, so far as concerns the presentation of scientific papers, as meetings of the Society. The Society will hereafter enjoy the possibly unique distinction of holding almost simultaneous meetings in different cities. The Chicago Section will retain its identity unchanged as regards sectional or local matters.

The following papers were read at the October meeting:

(1) Dr. G. M. GREEN: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface."

(2) Dr. G. M. GREEN: "One-parameter families of curves in the plane."

(3) Professor EDWARD KASNER: "The classification of analytic curves in conformal geometry."

(4) Mr. G. H. GRAVES: "Systems of algebraic curves of least order of genera 3 and 4."

(5) Mr. A. A. BENNETT: "Quadri-quadric transformations."

(6) Mr. A. A. BENNETT: "A set of postulates for a general field admitting addition, multiplication, and an operation of the third grade."

(7) Dr. T. H. GRONWALL: "On analytic functions of several variables."

(8) Mr. H. GALAJIKIAN: "Concerning the continuity and derivatives of the solution of a certain non-linear integral equation."

(9) Dr. G. M. GREEN: "On the limit of the ratio of arc to chord at a point of a real curve."

(10) Professor W. H. ROEVER: "Geometric derivation of a formula for the southerly deviation of falling bodies."

In the absence of the authors, the papers of Mr. Bennett and Professor Roever were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Darboux has founded a very elegant theory of conjugate nets on a curved surface, by identifying this theory with that of the Laplace transformations of a certain partial differential equation of the second order. In his first paper, Dr. Green associates with what is essentially Darboux's equation a second partial differential equation of the second order, obtaining in this way a completely integrable system of two differential equations. He is thus enabled, by following Professor Wilczynski's general procedure, to set up a purely projective theory of conjugate nets. Geometrically, this theory is evidently equivalent to that of a single one-parameter family of curves, since this family determines its conjugate family. Analytically, however, the determination of the conjugate family requires the integration of a partial differential equation of the first order. Supposing this to have been effected, the theory of a one-parameter family of curves becomes that of a conjugate net; we may say that by a supposed integration we have thrown into a canonical form the system of partial differential equations which define the one-parameter family as a component of a general net (not conjugate) on a curved surface. The use of the canonical form, however, is shown to be no restriction in generality, since it is proved that the invariants and covariants of the canonical system of differential equations are expressible explicitly in terms of the coefficients and variables of the more general system of differential equations, and this without requiring the integration which determines the family of curves conjugate to the given family. The theory as thus set up is put into relation with Professor Wilczynski's theory of congruences (with which it is closely connected), and with his theory of surfaces when referred to their asymptotic curves.

2. In his memoir in the *Transactions*, volume 12, entitled "One-parameter families and nets of plane curves," Professor Wilczynski founded a projective theory of nets in the plane

which Dr. Green employs in his second paper to study a single one-parameter family of plane curves. To determine projective properties of one family of curves, the net of which it is a component family must be completely and uniquely determined by it. For space curves the second component family may be taken as the conjugate family of curves; for the plane, however, no projectively intrinsic relation like conjugacy is immediately obvious. Dr. Green associates with the given one-parameter family a particular second family of curves, and obtains a canonical net completely determined by the given family. He is thus enabled to study the one-parameter family through this canonical net, and in particular to find a covariant triangle, referred to which the one-parameter family of curves is given in the neighborhood of a point by the following canonical development in non-homogeneous coördinates:

$$x = t + \tfrac{1}{2}y^2 + I^{(04)}y^4 + \cdots + \{I^{(21)}t^2 + I^{(31)}t^3 + \cdots\}y + \{I^{(12)}t + I^{(22)}t^2 + \cdots\}y^2 + \{I^{(13)}t + \cdots\}y^3 + \cdots,$$

in which the $I^{(jk)}$'s are absolute invariants of the one-parameter family of curves determined by the parameter t.

3. In his preceding contributions to conformal geometry, Professor Kasner considered only analytic arcs which were both real and regular. Any such arc can be reduced conformally to the axis of reals. He now studies the most general case, the arc being real or imaginary, and regular or irregular at the point considered. The fundamental group of course contains imaginary as well as real conformal transformations. In the regular case the arc can be reduced to one of three forms, $y = 0$, $y = ix$, $y = ix + x^n$. The classification of irregular arcs is much more complicated, since such an arc in general possesses absolute invariants, which may even be infinite in number. The theory of pairs of arcs (curvilinear angles), from this general point of view, is also outlined.

4. Following a plan devised by Castelnuovo and applied by Ferretti to compute linear systems of algebraic curves of genera 0, 1, and 2, Mr. Graves finds the systems of least order to which regular systems of genera 3 and 4 can be reduced by Cremona transformations. There are five types for genus 3 and eight for genus 4.

5. In this paper Mr. Bennett considers the properties of two-two algebraic transformations in an arbitrary field. In the cases of the real and the complex number fields, a quadri-quadric relation may be considered as a form of the addition theorem of an elliptic function and many of the properties of elliptic functions are more or less readily derivable from the quadri-quadric relations generated by them. Theorems derived from the quadri-quadric relation by means of transcendental processes, such as integration, are not necessarily applicable to finite fields. In this paper, the author confines himself entirely to rational processes, and yet obtains explicit formulas of closure (for Steiner and Poncelet polygons) and analogous theorems not previously stated. Every quadri-quadric transformation has certain other quadri-quadric transformations associated with it, several types of which are here given for the first time. By the use of these associated transformations some theorems already proved algebraically are here demonstrated in a much simpler manner.

6. One may consider in addition to the familiar operations of the first and second grades, viz., addition and multiplication respectively, an operation of the third grade. The result of this operation upon a and b, any two numbers, will be defined as $e^{(\log a)\cdot(\log b)}$. This operation is always uniquely possible when we confine ourselves to real positive numbers. One may define an analogous operation in the case of Galois fields, of certain matrix algebras, in the field of all analytic functions, etc. A large body of theorems is common to all such fields, thus in every "field of the third grade," there is a unique zero, a unique unity, addition and multiplication are both commutative, etc., but in general mathematical induction is not applicable. In this paper Mr. Bennett gives a set of postulates for a field of the third grade, and without the use of mathematical induction proves the principal elementary theorems which hold in all such fields.

7. Cousin has stated (*Acta Mathematica*, volume 19) that when an analytic function $f(x_1, \cdots, x_n)$ is uniform and meromorphic for x_1 inside a domain S_1 in the x_1-plane, $\cdots$, x_n inside a domain S_n in the x_n-plane, then this function may be expressed as a quotient of two uniform functions holomorphic in $S_1, \cdots, S_n$ and without common zero manifolds of higher dimension than $2n - 4$.

Dr. Gronwall shows that Cousin's proof is not valid unless all, or all but one, of $S_1, \cdots, S_n$ are simply connected, and constructs an example contradicting the theorem when this condition is not fulfilled.

It may be proved, however, that if the condition on the common zeros is dropped, $f(x_1, \cdots, x_n)$ may be expressed as a quotient of two uniform and holomorphic functions having common zero manifolds of $2n - 2$ dimensions, and these common zeros cannot in general be removed without destroying the uniformity of the functions.

8. In this paper Mr. Galajikian presents two theorems concerning the non-linear integral equation

$$y(x, x_0) = g\left\{x, \int_{x_0}^{x} f_1(x, t, y(t, x_0))dt, \int_{x_0}^{x} f_2(x, t, y(t, x_0))dt\right\}.$$

The first of these theorems states that if this equation has a solution under certain definite conditions, then this solution is continuous in the arguments x and x_0. The second theorem states that the solution has first derivatives with respect to the same arguments. In both cases the proofs consist in showing that the usual definitions may be satisfied under proper conditions.

The methods apply to the more general case of n such equations with m unknown functions.

9. In his third paper, Dr. Green considers the limit of the ratio of arc to chord at a point on a real curve whose equation is $y = f(x)$. Taking the origin at the point under consideration, the arc is assumed to be given by

$$s(x) = \int_0^x \sqrt{1 + y'^2}dx.$$

The chord is $\sqrt{x^2 + y^2}$. If the curve have a definite tangent at the origin, it is easily found that a limit of the ratio of arc to chord exists if and only if $s(x)$ have a derivative $s'(+0)$ from the right at the origin. The limit is then $s'(+0)/\sqrt{1 + y_0'^2}$, and is unity if and only if $s'(+0) = \sqrt{1 + y_0'^2}$, in particular if the function y'^2 is continuous at the origin. An example is, however, constructed in which

the limit is unity although y'^2 is discontinuous at the origin. The whole discussion is closely related to Hahn's well-known work on the differentiation of the integrals of pointwise discontinuous functions.

10. Professor Roever has already derived by two different methods the formula*

$$\text{(1)} \qquad \text{S. D.} = \frac{1}{6}\left[2\omega^2 \sin 2\phi_0 + 5\left(\frac{\partial g}{\partial x}\right)_0 \right]\frac{h^2}{g_0},$$

where S. D. stands for southerly deviation, h for height through which the body falls, ω for angular velocity of earth's rotation, g_0 and ϕ_0 for the values of the acceleration and astronomical latitude, respectively, at the point P_0 from which the body falls, and $(\partial g/\partial x)_0$ for the value at P_0 of the derivative of g along the meridian to the north.

Let P_1 be a point fixed with respect to the earth, and P_0 a point above P_1 and in the normal at P_1 to the level surface through P_1. Then P_1 lies in the curve d which passes through P_0 and is the locus of the bobs of all plumb-lines supported at P_0. The path, with respect to the earth, of the body which falls from P_0 is a curve c, the orthographic projection of which on the meridian plane of P_0 is the curve c'' which pierces the level surface of P_1 in the point C''. Then S. D. $= P_1C''$.

The curves d, c (and therefore also c'') and the line of force of the weight field which passes through P_0 are all tangent to the normal at P_0 of the level surface through P_0. Hence if we put $h = P_0P_1$, we have

$$\text{(2)} \qquad \text{S. D.} = P_1C'' = \left(\frac{1}{2}\frac{1}{\rho_d} - \frac{1}{2}\frac{1}{\rho_{c''}}\right) h^2,$$

where $1/\rho_d$ and $1/\rho_{c''}$ are the curvatures at P_0 of d and c'' respectively.

The curvature at P_0 of d is twice that of the line of force of the weight field of force which passes through P_0.

The curve c'' osculates at P_0 a curve which may be regarded as the path of a particle starting from rest at P_0, in a positional field of force whose potential function is simply related to that of the weight field. Hence the curvature at P_0 of c'' is one third that of the line of force of this posi-

* See *Transactions*, vol. 12, No. 3; vol. 13, No. 4.

tional field which passes through P_0. (See Kasner's Princeton Colloquium Lectures, page 9, second footnote.)

By a well-known theorem the curvature of a line of force of the weight field is the logarithmic derivative of g in the direction on a level surface in which g increases most rapidly, i. e.,

$$\frac{\partial}{\partial x}\log g = \frac{\partial g/\partial x}{g}.$$

From the above relations

$$(3)\quad \frac{1}{2\rho_d} = \left(\frac{\partial g}{\partial x}\right)_0 \Big/ g_0, \quad \frac{1}{2\rho_{c''}} = \frac{1}{6g_0}\left[\left(\frac{\partial g}{\partial x}\right)_0 - 4\omega^2 \sin\phi_0 \cos\phi_0\right],$$

and hence by (2) we get (1).

For the data: $h = 49{,}024$ cm., $\phi_0 = 45°$, and for the potential function for which the Bessel ellipsoid is a level surface and the formula of Helmert gives the acceleration,

$$g_0 = 980.6, \quad \left(\frac{\partial g}{\partial x}\right)_0 = 8.1568 \times 10^{-9}, \quad \omega^2 = 5.3173 \times 10^{-9},$$

and hence by formula (1)

$$\text{S. D.} = +\ .021 \text{ cm.}$$

F. N. COLE,
Secretary.

THE TWENTY-FOURTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-fourth regular meeting of the San Francisco Section of the Society was held at Stanford University on October 25, 1913. Twenty-one persons were present, including the following members of the Society:

Professor R. E. Allardice, Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor G. I. Gavett, Professor Charles Haseman, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, and Professor E. W. Ponzer.

Professor Edwards, chairman of the section, presided at the opening of the morning session, Professor Manning, chairman-elect, then took the chair. The following officers of the Section were elected for the ensuing year: chairman, Professor Manning, secretary, Dr. Buck; program committee, Professors Blichfeldt, Lehmer, and Dr. Buck.

It was voted to hold the spring meeting at Seattle provided the Pacific Coast Association of Scientific Societies meets there, and the fall meeting at the University of California on October 24, 1914. A committee, consisting of Professors Haskell, Allardice, Blichfeldt, Lehmer, and Noble, was appointed to make arrangements on the part of the Section for the summer meeting of the Society at San Francisco in 1915.

The members present lunched together between sessions.

The following papers were presented at this meeting:

(1) Professor L. M. HOSKINS: "Note on the motion of a freely falling body."

(2) Professor M. W. HASKELL: "Second note on the Del Pezzo quintic."

(3) Mr. B. A. BERNSTEIN: "A complete set of postulates for the logic of classes expressed in terms of the operation 'exception,' and a proof of the independence of a set of postulates due to Del Ré."

(4) Professor M. W. HASKELL: "On the singularities of twisted curves."

(5) Professor W. C. EELLS: "Number systems of the North American Indians."

(6) Professor G. A. MILLER: "Some properties of the group of isomorphisms of an abelian group."

(7) Professor W. A. MANNING: "On the class of doubly transitive groups."

In the absence of the authors, the papers of Professors Eells and Miller were read by title. Abstracts of the papers follow below.

1. One of the simplest methods of studying the motion of a freely falling body is to refer the motion to axes fixed in the earth, making use of the relation between the accelerations of a particle referred to fixed and moving axes. The equations for the motion relative to the earth may thus be formed by introducing, in addition to the actual forces, the fictitious "centrifugal" and "compound centrifugal" forces. The

former may be taken account of by adding to the earth's attraction potential V a potential $U = \frac{1}{2}\omega^2\rho^2$, ω denoting the earth's angular velocity, and ρ distance from the axis of rotation. For a body falling from rest the effect of the compound centrifugal force is easily computed to a close approximation, the main effect being an easterly displacement proportional to the cube of the time of falling; while a very minute secondary effect would be a deviation toward the equator, proportional to the fourth power of the time. These displacements are superposed upon those due to the field of force corresponding to the potential $V + U$. The question whether this field would cause a deviation northward or southward from a plumb-line suspended from the point where the body starts, is closely connected with the question of the curvature of the lines of force in this field; the path would lie on the convex side of that line of force passing through the position of rest, while the plumb-line would hang on the concave side. So far as the effect of this field is concerned, the body will fall on that side of the plumb-line toward which the lines of force are convex. Analysis indicates that the convexity is toward the equator.

A matter of interest in the rigorous analysis is the fact that the motion in the meridian plane is identical with the plane motion of a particle in a stationary field, of force of potential $V - U\rho_0^4/\rho^4$, ρ_0 being the initial value of ρ.

The analysis is easily extended to the case in which the body has an initial velocity.

2. This paper of Professor Haskell contains the complete form of the Hessian of the given quintic and its reduction to a form which shows that the 5 cusps and the 5 inflexions of the quintic are the complete intersection of the quintic with a certain covariant cubic.

3. Mr. Bernstein presents a set of six postulates all expressed in terms of the single undefined operation "exception,"—the operation which forms from the two logical elements a, b the element ordinarily denoted by $a\bar{b}$ and read, "a except b" or "a which is *not* $= b$." The postulates are proved to be consistent, independent of one another, and "sufficient" for the logic of classes.

The writer also proves the independence of a set of postulates for the algebra of logic due to Professor Del Ré.

4. In this paper Professor Haskell considers the generalization of a theorem due to Clebsch on the singularities of plane curves when given in parameter form, and gives simple criteria for the existence and enumeration of the various singularities of twisted curves.

5. Professor Eells' paper is based upon a study of the number systems of over 300 Indian languages of North America. The evidence for and against a digital origin of counting is presented; the use of additive, subtractive, multiplicative, divisive, and duplicative principles in the formation of number systems is analyzed; examples of decimal, vigesimal, quinary, octonary, quaternary, and ternary systems of numeration are given and discussed; and other noteworthy features of primitive number systems are mentioned. The paper will appear serially in the *American Mathematical Monthly*.

6. As the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, Professor Miller confined his attention to a consideration of the group of isomorphisms I of an abelian group G of order p^m, p being a prime number. After determining the number of the Sylow subgroups of order p^m contained in I, it was proved that each of these Sylow subgroups is transformed into itself under I by a group of order $p^m(p-1)^\lambda$, where λ is equal to the total number of the invariants of G. This group of order $p^m(p-1)^\lambda$ contains an abelian subgroup of order $(p-1)^\lambda$, which is the direct product of λ cyclic subgroups of order $p-1$.

A necessary and sufficient condition that I contains only one subgroup of order p^m is that no two of the invariants of G are equal to each other. Whenever G contains at least two equal largest invariants then I contains an operator whose order is equal to these invariants. If the number of each set of equal invariants of G is less than p, then the group of isomorphisms of G cannot involve any operator whose order is a power of p and exceeds the largest invariant of G. When the Sylow subgroups of order p^m are abelian, either G must be cyclic or m must be less than 3.

7. If n is the degree and u the class of a doubly transitive group, Bochert's theorem asserts that u is greater than $\frac{1}{3}n-$

$\frac{2}{3}\sqrt{n}$. This is based upon the fact that two non-commutative substitutions of degree u cannot have less than $\frac{1}{3}u$ letters in common. Professor Manning called attention to the case in which at least one of the substitutions of degree u in the group is of order 2. Here two non-commutative substitutions of degree u have at least $\frac{1}{2}u$ letters in common, from which he concludes that u is greater than $\frac{1}{2}(n - \sqrt{n}) - 1$.

THOMAS BUCK,
Secretary of the Section.

THE SEVENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE seventh regular meeting of the Southwestern Section of the Society was held at the University of Missouri, Columbia, Mo., on Saturday, November 29, 1913. About twenty-five persons attended the meeting, including the following sixteen members of the Society:

Professor L. D. Ames, Professor C. H. Ashton, Dr. Henry Blumberg, Professor W. C. Brenke, Professor E. W. Davis, Dr. E. L. Dodd, Dr. Otto Dunkel, Professor E. R. Hedrick, Professor Louis Ingold, Professor O. D. Kellogg, Dr. A. J. Kempner, Professor W. H. Roever, Professor H. E. Slaught, Professor J. N. Van der Vries, Miss Eula Weeks, Professor W. D. A. Westfall.

The morning session opened at 10.30 A.M. and the afternoon session at 2 P.M. Professors Hedrick and Slaught presided. It was decided to hold the next meeting of the Section at the University of Nebraska on November 28, 1914. The following programme committee was elected: Professor E. W. Davis (chairman), Dr. S. Lefschetz, Professor O. D. Kellogg (secretary). Those present attended a smoker at the house of Professor Kellogg on the evening before the meeting.

The following papers were presented:

(1) Professor W. C. BRENKE: "An example of Abel's integral equation with discontinuous solution."

(2) Professors E. R. HEDRICK and LOUIS INGOLD: "Generalization of Taylor's series."

(3) Dr. S. LEFSCHETZ: "Double integrals of the third kind attached to an algebraic variety."

(4) Professors E. R. HEDRICK and W. D. A. WESTFALL: "Jacobians and existence theorems for implicit functions."

(5) Professor O. D. KELLOGG: "Sign-changes in functions of an orthogonal set."

(6) Dr. E. L. DODD: "The weighting of measurements on the basis of their relative magnitudes."

(7) Dr. A. J. KEMPNER: "On irreducible equations."

(8) Miss EULA WEEKS: "Note on the enclosable property."

(9) Dr. HENRY BLUMBERG: "On an extension of Baire's fundamental theorem concerning functions representable as the limit of a sequence of continuous functions."

(10) Dr. HENRY BLUMBERG: "On the oscillation function and related functions."

(11) Mr. A. R. SCHWEITZER: "On a system of four dimensional simplexes inscribed in a hypersphere."

In the absence of the authors, Dr. Lefschetz's paper was presented by Professor Van der Vries, and Mr. Schweitzer's was read by title. Abstracts of the papers follow.

1. The problem of determining the quantity of flow through a weir notch so that the flow shall be proportional to the depth of liquid in the notch is solved by finding $f(x)$, the form of the notch, from the equation $\int_0^h \sqrt{h-x}\, f(x)dx = kh$, where h is the depth of liquid in the notch. This equation reduces upon differentiation to Abel's integral equation with discontinuous solution. Professor Brenke solves it by means of a substitution of the form $f(x) = \varphi(x) + g(x)$, where $g(x)$ is continuous and $\varphi(x)$ is discontinuous.

2. In this paper, Professors Hedrick and Ingold point out that with a suitably extended notion of orthogonality of functions, developments in Taylor's series may be regarded as expansions in terms of the functions of an orthogonal system, and that many of the usual geometric analogies for such expansions hold also for ordinary Taylor expansions. The notion of orthogonality which is employed involves a set of operations as well as the set of functions under consideration, so that the orthogonality of a set of functions has no meaning until a corresponding set of operations is specified. With this point of view, the difference between a function

and its Taylor development is orthogonal, with respect to certain operations, to all of the functions 1, x, x^2, $\cdots$. This difference, as well as any function orthogonal to the set just given, may well have "Taylor" developments in terms of other functions, $\theta_0, \theta_1, \theta_2, \cdots$, that is, developments in which the coefficients are determined by operations similar to those used in Taylor's series.

3. It has been shown by Picard (Traité des Fonctions algébriques de deux Variables, volume 2, page 231) that there is a minimum ρ to the number of logarithmic singular curves which a simple integral of the third kind belonging to an algebraic surface may have. Dr. Lefschetz has shown the existence of a similar number ρ_v for an algebraic variety. In the present paper he shows that if a double integral of the third kind belonging to an algebraic variety of four dimensions has more than one transcendentally singular surface, it has at least $\rho_v + 1$ of them. It is interesting to note that double integrals, and in all probability multiple integrals, belonging to an algebraic spread do not give any new invariant number corresponding to ρ. It seems likely that Picard's formula for integrals of the second kind (loc. cit., page 409) can be extended with few changes.

4. In this paper Professors Hedrick and Westfall simplify and extend certain theorems in their paper "An existence theorem for implicit functions" read at the summer meeting of the Society. Various forms of difference jacobians are studied, together with their geometric interpretations. Applications are made to inverse transformations.

5. Orthogonality alone of a set of continuous functions does not insure the phenomenon one notices in the sets of orthogonal functions in common use, namely that when arranged in the order of the number of changes of sign, the nth function changes sign at least $n - 1$ times. A sufficient condition for this is the non-vanishing of the determinants $|\varphi_1(x_1), \varphi_2(x_2), \cdots, \varphi_n(x_n)|$ which arise in the interpolation problem. Professor Kellogg's paper calls attention to this fact, shows the condition to be satisfied in the case of a number of the commoner sets of orthogonal functions, and shows as a result that the difference between a function and the approximation

to it by means of the first n terms of its development in terms of a set of orthogonal functions satisfying the above condition, the coefficients being determined after the Fourier manner, changes sign at least $n - 1$ times.

6. In forming a weighted mean of measurements as the most acceptable value for the quantity measured, the weights are usually assigned on the basis of knowledge that some of the measurements are made under more favorable conditions than others. Dr. Dodd discusses the question as to whether the measurements themselves may not be made to yield weights of value in forming weighted means; for instance, should not the median, because of its position among the measurements bear a heavier weight? The "select weighted means" proposed in the paper are compared with the unweighted means from the point of view of reliability.

7. Assuming $\psi(z)$ to be an integral rational function of z with real rational coefficients, irreducible in the natural domain, Dr. Kempner proves a set of theorems on equations $\psi(z) = 0$ having at least one root of rational absolute value, and a corresponding set of theorems on equations $\psi(z) = 0$ having at least one root of rational real part. Although the proofs are of very elementary character, the results are believed to be new. The following theorems represent the type of results obtained. I. *a*. If $\psi(z) = 0$ has a complex root of absolute value 1, $\psi(z)$ is of even order and reciprocal. I. *b*. If $\psi(z) = 0$ has a purely imaginary root, $\psi(z)$ is of even order and does not contain odd powers of z. II. All roots of $\psi(z) = 0$ of rational absolute value (of rational real part) have the same rational absolute value (the same rational real part). III. All equations $\psi(z) = 0$ of degree $n < 10$ having at least one root of rational absolute value (at least one root of rational real part) are completely solvable by radicals. IV. No equation $\psi(z) = 0$ of odd order can have a complex root of rational absolute value (a complex root of rational real part).

8. Referring to Fréchet's letter to Hedrick (*Transactions*, volume 14, page 320), Miss Weeks shows that with the assumptions used by Fréchet, the definition of the enclosable property set up by him is identical with that proposed by Hedrick (*Transactions*, volume 12, page 289), in so far as it is a property of

the fundamental domain. The definitions are identical in the sense that if a set of assemblages of the kind required by Fréchet can be selected, then a set of assemblages satisfying Hedrick's conditions exists. The apparent slight difference in the two statements noticed by Fréchet is therefore only superficial.

9. Baire has shown that a necessary and sufficient condition that a function may be the limit of a sequence of continuous functions is that it be pointwise discontinuous in every perfect set. (Leçons sur les Fonctions discontinues, Paris, 1905, §§ 68, 73, 74 and 77). The proof is given for the case where the domain of the independent variable x is any perfect set. In the *Acta Mathematica*, volume 30, 1906, pages 1 to 48, Baire remarks that the proof applies also to the case where the domain of x is any closed set, and extends the theorem to the case where the domain of x is any set whatever (closed or open). The proof is long, and comparable in difficulty with that of the original theorem. Dr. Blumberg shows how this extension may be made in a very simple and immediate fashion.

10. Let $f(x)$ be any real, bounded, single-valued, continuous or discontinuous function of the real variable x. Let $\omega f(x)$ denote the oscillation of $f(x)$ at the point x, $M(x)$ the maximum at that point, and $m(x)$ the minimum at the point. Dr. Blumberg's paper contains the following results: a necessary and sufficient condition that a function may be an oscillation function; a necessary and sufficient condition that a function may be the oscillation function of an oscillation function; a new proof, by means of the above, of Sierpinski's theorem that $\omega\omega f(x) = \omega f(x)$; the establishment of the inequalities $\omega M(x) \leqq \omega\omega f(x)$ and $\omega m(x) \leqq \omega\omega f(x)$; the proof by means of these inequalities that if $\omega f(x)$ is continuous, both $M(x)$ and $m(x)$ are continuous; the proof that a necessary and sufficient condition that $f(x)$ may be pointwise discontinuous is that $\omega\omega f(x) = \omega f(x)$; a method for finding the complete solution of the equation $\omega f(x) = c(x)$, where $c(x)$ is a given continuous function and $f(x)$ is to be found, with a discussion of more general equations of the same type; indications of extensions of the preceding results to functions of several variables defined in more general domains.

11. In the *American Journal of Mathematics*, volume 34 (1912), page 173, Mr. Schweitzer has shown how to generate a quasi-four-dimensional geometry ${}^4R_4^{(0)}$ by adjoining a point tactically to the system ${}^3R_3$ and assuming the axiom "$\alpha R\beta\gamma\delta\epsilon$ implies $\delta R\epsilon\alpha\beta\gamma$" which ensures that the generating relation is alternating. The resulting system is sufficient for the usual three-dimensional projective geometry if an axiom expressing Dedekind continuity (suitably modified for projective geometry) is added. This geometry ${}^4R_4^{(0)}$ may be regarded as underlying a system of four-dimensional simplexes inscribed in a hypersphere. In the *Archiv der Mathematik und Physik*, volume 21 (1913), page 204, E. Study has remarked that the figure of five ordered real points, no four of which are coplanar, has a (single) property, "signatur" (+) or (−), which is not disturbed by positive real collineations. It seems simpler and altogether more convenient to regard Study's figure of five points with positive or negative "signatur" as a sensed simplex in quasi-four space as indicated above.

O. D. KELLOGG,
Secretary of the Section.

THE INFINITE REGIONS OF VARIOUS GEOMETRIES.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, September 8, 1913.)

MOST geometers are now conscious that the introduction of points at infinity in such a way that in plane geometry they form a line, in three-dimensional geometry a plane, is, to a large extent, an arbitrary convention; but few of them would probably admit that this remark has much practical importance (except in so far as they might regard any question concerning the logical foundation of geometry as having practical importance) since the convention here referred to is commonly regarded as being the only desirable one. It is the object of the present paper to point out more explicitly and in greater detail than has, to my knowledge,

been done before that this view is by no means justified.* For investigations in the realm of projective geometry, the conception of the line or the plane at infinity is the natural and appropriate one; and inasmuch as every geometric investigation can be forced into the projective mould, this conception of the infinite region can always be made to do duty. It is, however, not the most desirable one in all cases.

For the sake of simplicity I shall consider first merely plane geometry, and, indeed, the geometry of the *real* plane.

§1. *The Projective Plane.*

In projective geometry we have to deal with the 8-parameter group of point transformations

$$(1)\qquad X' = \frac{a_1X + b_1Y + c_1}{a_3X + b_3Y + c_3}, \qquad Y' = \frac{a_2X + b_2Y + c_2}{a_3X + b_3Y + c_3}, \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

where X, Y stand for non-homogeneous cartesian coordinates, which we will suppose to be rectangular. Every transformation of this group carries over points, in general, into points and collinear points into collinear points. Strictly speaking, the points on the line

$$a_3X + b_3Y + c_3 = 0$$

are not transformed at all; but the nearer one comes to such a point the farther away will the transformed point lie, and we consequently speak of these points as being "thrown to infinity," and thus introduce the conception of points at infinity in projective geometry. All these points at infinity, or *ideal points*, we regard as lying on a single line in order not to have any exception to the statement that collinear points go over into collinear points. We then introduce

*This article was already in type before I saw a paper by H. Beck, *Archiv d. Math. u. Physik*, ser. 3, vol. 18, p. 43, which has much the same tendency. In that paper only the space of inversion is considered, and the questions taken up are in the main different from those here treated. The author seems not to have noticed that in my book of 1894 (cited below) I had explicitly described the infinite region for the complex plane of inversion, and implicitly (by the constant use of the term "Null sphere at infinity" with its equation) for spaces of higher dimensions.

homogeneous coordinates $(x : y : t)$ by means of the equations

$$X = \frac{x}{t}, \qquad Y = \frac{y}{t},$$

in order to be able conveniently to designate the various points at infinity. All this is so well known that the foregoing brief sketch will suffice.

If we now approach other groups of transformations with an open mind and in the same spirit in which we just approached the group (1), we shall be led in the same way to other geometries worthy of being considered side by side with projective geometry,* and each of these geometries will have just as much right to its own peculiar infinite region as has projective geometry.

§ 2. *The Plane of Analysis.*†

We shall consider in this section the geometry which is based on the 6-parameter group

$$(2) \qquad \begin{aligned} X' &= \frac{\alpha_1 X + \beta_1}{\gamma_1 X + \delta_1} \qquad \begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{vmatrix} \neq 0, \\ Y' &= \frac{\alpha_2 Y + \beta_2}{\gamma_2 Y + \delta_2} \qquad \begin{vmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{vmatrix} \neq 0, \end{aligned}$$

where again, for the sake of concreteness, we assume all the quantities involved to be real.

Suppose, first, that neither γ_1 nor γ_2 is zero. Then (2) gives a transformation of the finite plane which carries over every point into a new position except that the points on the lines

$$(3\text{–}4) \qquad \gamma_1 X + \delta_1 = 0, \quad \gamma_2 Y + \delta_2 = 0$$

are not transformed. If we take any point (X_1, Y_1) on one of these lines, and allow another point (X_2, Y_2) to approach it in such a way as never to lie on the line in question, then the image of this second point recedes to infinity. In this sense we may say that the lines (3), (4) are both thrown to infinity.

* Cf. Klein's Erlanger Programm (1872), where, however, the point with which we are here concerned, the character of the infinite region, is not brought out.

† Cf. Osgood, *Transactions*, vol. 13 (1912), p. 159.

Similarly, since the inverse of (2) is

$$X = \frac{-\delta_1 X' + \beta_1}{\gamma_1 X' - \alpha_1}, \qquad Y = \frac{-\delta_2 Y' + \beta_2}{\gamma_2 Y' - \alpha_2},$$

we see that there are also two lines

$$\gamma_1 X' - \alpha_1 = 0, \quad \gamma_2 Y' - \alpha_2 = 0 \tag{5–6}$$

which come from infinity.

If one but not both of the quantities γ_1, γ_2 are zero, only one of the lines (3), (4) will go to infinity (the other equation no longer representing any locus), and only one of the lines (5), (6) will come from infinity; while if $\gamma_1 = \gamma_2 = 0$, there are no exceptional lines of this sort.

Apart from these exceptional lines, every line $x = \text{const.}$, and also every line $y = \text{const.}$, goes over into another line of the same form; and every line of one of these forms comes from another such line. These lines we shall find play a fundamental rôle in the kind of geometry we are now considering, and it will be convenient to designate them as *ground-lines*, distinguishing between two *kinds* of ground-lines according as the equation is $x = \text{const.}$ or $y = \text{const.}$ We may then say

The transformations (2) *are one-to-one point transformations which carry every ground-line into a ground-line of the same kind.*

This statement, however, is at present not accurate since it admits certain exceptions. It becomes accurate only when, as we shall now do, we introduce points at infinity (ideal points) in such a way that they form two new ideal ground-lines, one of each kind. This may be easily effected by synthetic methods. We will, however, proceed analytically by the use of homogeneous coordinates defined by the equations

$$X = \frac{x_1}{x_2}, \qquad Y = \frac{y_1}{y_2}.$$

These, it should be noticed, are not the homogeneous coordinates of projective geometry, a point being determined here by *four* coordinates $(x_1 : x_2,\ y_1 : y_2)$ involving, however, only two ratios. The finite points are those for which neither x_2 nor y_2 is zero, while the points at infinity are, by definition, sets of coordinates in which either x_2 or y_2 or both are zero,

but where not both the x's (and also not both the y's) are zero. These last mentioned sets of coordinates we do not speak of as being points at all. Moreover we agree, that two points at infinity shall be identical when and only when the x's of one are proportional to the x's of the other, and also the y's of one are proportional (though perhaps with a different factor of proportionality) to the y's of the other. This convention is the natural one to make, since finite points will be identical under, and only under, the same conditions.

Every homogeneous equation of the first degree in y_1, y_2 represents a line parallel to the axis of x (an x-ground-line) except the equation $y_2 = 0$ which is satisfied only by ideal points and which we speak of as the x-ground-line at infinity.

Similarly the equation $x_2 = 0$ gives us the y-ground-line at infinity.

These two ideal ground-lines obviously contain all points at infinity. They have one and only one point in common, namely $(1 : 0, 1 : 0)$, which we speak of as the double point at infinity in distinction to the infinite number of simple points at infinity where only one of the two quantities x_2, y_2 is zero.

The transformation (2) may be written

$$(7)\qquad \begin{aligned} \rho x_1' &= \alpha_1 x_1 + \beta_1 x_2, \quad \sigma y_1' = \alpha_2 y_1 + \beta_2 y_2, \quad \alpha_1\delta_1 - \beta_1\gamma_1 \neq 0, \\ \rho x_2' &= \gamma_1 x_1 + \delta_1 x_2, \quad \sigma y_2' = \gamma_2 y_1 + \delta_2 y_2, \quad \alpha_2\delta_2 - \beta_2\gamma_2 \neq 0. \end{aligned}$$

We pass now to straight lines other than ground-lines. Such a line may be written

$$AX + BY + C = 0,$$

where neither A nor B is zero; or, in homogeneous coordinates,

$$Ax_1y_2 + By_1x_2 + Cx_2y_2 = 0.$$

This is a special case of the general bilinear equation

$$(8)\qquad ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2 = 0.$$

This equation is readily seen to represent, in general, an equilateral hyperbola with ground-lines as asymptotes. It may, however, as we have just seen represent a single straight line, not a ground-line, or it may represent any pair of ground-

lines, one of each set. All these curves we shall speak of as bilinear curves; the last case, which is characterized algebraically by the vanishing of the determinant* $ad - bc$, being designated as a singular bilinear curve. *A non-singular bilinear curve is a straight line or a hyperbola according as it does or does not pass through the double point at infinity.* It should be noticed, however, that a single ground-line is not a bilinear curve, but is something simpler.

It is clear that all transformations (7) carry bilinear curves into bilinear curves leaving them singular or non-singular as the case may be. All non-singular (and also all singular) bilinear curves are equivalent under the group (7).

It will thus be seen that in this kind of geometry the classification of curves as straight lines, conics, etc., is an artificial one; some straight lines being, for instance, ground-lines and some bilinear curves. This fact becomes still clearer if we consider some of the fundamental properties of the various curves.

Two distinct ground-lines of the same kind have no point in common. For two such y-ground-lines are given by the equations

$$\begin{aligned} p_1x_1 + q_1x_2 &= 0 \\ p_2x_1 + q_2x_2 &= 0 \end{aligned} \qquad p_1q_2 - p_2q_1 \neq 0,$$

which are satisfied only by $x_1 = x_2 = 0$. This we do not speak of as a point.

Two ground-lines of different kinds have one and only one point in common. In particular, two different finite, x-ground-lines have different points at infinity, namely the points where they meet the y-ground-line at infinity. It will not do in this case to say that parallel lines meet at infinity.

A ground-line and a non-singular bilinear curve always have one and only one point in common. A general algebraic proof may readily be given, or we may first reduce the ground-line (which we may suppose to be a y-line) and the bilinear curve to the normal forms

$$x_1 = 0, \quad x_1y_1 + x_2y_2 = 0$$

by a transformation (7), as is clearly always possible. These

* We note in passing that this determinant is an invariant (combinant) of the bilinear curve under the transformation (7).

curves have obviously the point $(0 : 1, 1 : 0)$, and no other point, in common.

Two distinct non-singular bilinear curves have not more than two points in common. If there is any point of intersection, let P be such a point, and throw it to the double point at infinity by a transformation (7). Then the bilinear curves become straight lines, and these can intersect at most in one point besides the double point at infinity.

It is readily seen from this proof, or otherwise, that the case of a single point of intersection may be regarded as the limit of cases in which the curves intersect in two points and where these two points approach each other. We say, then, in this case that the curves are tangent to each other. It is readily seen that this definition of contact coincides with the ordinary one when the point of contact is finite. Contact at infinity, on the other hand, means something very different here and in projective geometry. This is illustrated by the following facts:

A ground-line cannot touch a bilinear curve, since it *always* meets it in only one point. In particular, an equilateral hyperbola whose asymptotes are ground-lines must not now be regarded as being tangent to these ground-lines at infinity.

Two parallel straight lines, not ground-lines, must be regarded as non-singular bilinear curves which have contact at the double point at infinity.

Two bilinear curves (equilateral hyperbolas) with one asymptote in common do not in general have contact with one another at infinity. They do this, however, if they not only have a common asymptote but are of the same size without coinciding.

§ 3. *Correspondence of the Plane of Analysis to a Ruled Quadric.*

The two sets of ground-lines of the plane of analysis strongly suggest the two sets of rulings on a non-singular quadric surface. In each case every line of one set meets every line of the other set, but meets no other line of its own set. That this is not merely a superficial analogy may be seen as follows:

Let S denote any unparted hyperboloid or hyperbolic paraboloid in *projective* space, and designate one of the two sets of rulings arbitrarily as the first set, the other as the second set. It is well known that the cross-ratio of the range

in which four rulings of one set are cut by a ruling of the other set is independent of the choice of this ruling, and that this cross-ratio is called the cross-ratio of the four rulings. Now let us choose three fixed rulings of the first set and attach to every ruling of this set a value of the parameter X equal to the cross-ratio which this ruling makes with the three fixed rulings. In this way every ruling of the first set is uniquely determined by a value of X, the three originally selected rulings corresponding to the values $X = 0, 1, \infty$. Similarly we determine the rulings of the second set by the values of a parameter Y. Since every point on S is determined as the intersection of one ruling of the first with one of the second set, we have thus a system of coordinates (X, Y) between whose real values and the real points on S, finite or at infinity, there is a continuous one-to-one correspondence, provided we include the values $X = \infty$ and $Y = \infty$. We thus establish a continuous one-to-one correspondence between the points of S and the points of the plane of analysis of such a sort that to the two sets of rulings of S correspond the two sets of ground-lines of the plane of analysis.* Moreover the transformations (2) of the plane of analysis evidently correspond to the transformations of S into itself which carry over every ruling of each set into another ruling of the same set in such a way that each set of rulings is transformed *projectively* into itself. Such transformations are yielded by those collineations of space which carry over S into itself without interchanging the two sets of rulings; and conversely it may be proved that every transformation of S into itself, of the kind we want, may be obtained in this way. Thus we may say:

Abstractly considered, the real plane of analysis with its real transformations (2) *is identical with a real, non-singular, ruled quadric S and such of its real collineations into itself as do not interchange the rulings of the two sets.* If, however, we wish to get an intuitional substratum for this abstract theory, the plane of analysis would seem to be the simpler from every point of view except that it is, at first, a little unfamiliar.†

Several generalizations of what has just been said will at once suggest themselves.

* By taking for S a hyperbolic paraboloid, we can, if we wish, make the infinite region of the plane of analysis correspond to the infinite region of S.

† The quadric surface S also has its peculiar advantages owing to the way it is immersed in three-dimensional space.

In the first place, no difficulty is introduced by allowing all the quantities considered to be complex. In this way we get the complex plane of analysis, on the one hand, and the complex surface S, on the other; that is, not merely the real points of S, which we have considered exclusively so far, but also its complex points. Moreover, there is no necessity now to assume that S has real rulings, or even that it has real points: it may be any non-singular quadric.*

On the other hand, either in the real or in the complex plane of analysis, we may consider not the group of transformations (2), but the larger group consisting of these transformations (which we will now call the *direct* transformations) and also of the *indirect* transformations obtained by combining them with the transformation

$$X' = Y, \quad Y' = X.$$

This larger group obviously corresponds to the group of *all* the collineations of S into itself. From this point of view we may say:

The geometry of analysis in the plane, and the projective geometry on any non-singular quadric surface, are abstractly identical.

§ 4. *The Plane of Inversion.*

This last theorem recalls one of Klein's early discoveries† namely that the geometry of inversion in the plane is abstractly identical with the projective geometry on any non-singular quadric surface. By combining these two facts we get the result:

The geometry of analysis in the plane and the geometry of inversion in the plane are abstractly identical.

This result, like the two from which we have deduced it, is true, however, only if we disregard all questions of reality. Let us, then, go back and consider the geometry of inversion for its own sake, at first in a very elementary manner.

We start from the real finite plane and its rectangular cartesian coordinates (X, Y), and consider the 6-parameter group of transformations generated by all rigid motions

* We mention in passing that the bilinear curves may readily be shown to correspond to the plane sections of S, a fact which is, of course, equally true if we restrict ourselves to reals.

† *Math. Annalen*, vol. 5 (1872), p. 267.

combined with inversions in circles and reflections in lines. Since an inversion throws the points near the center of the circle to a great distance, and brings the points at a great distance into the neighborhood of the center, we are in the habit of saying that it throws the center to infinity, and vice versa; and since, except for this, we have a one to one transformation, we are here led, in precisely the manner of §§ 1, 2, to introduce the conception of a *single* point at infinity, and to say: *In the geometry of inversion of the real plane, the infinite region shall be regarded as forming a single point.* This point of view is so familiar, through its importance in the theory of functions of a single complex variable, that we need not insist on it further. It is, indeed, the *one* case in which it is generally conceded, though often grudgingly, that the conception of the *line* at infinity is not the only possible one. We recall that the simplest class of curves here is that which consists of all straight lines and all circles (we consider here only real loci), these curves being carried into one another by the transformations of the group. Straight lines differ from circles merely in that they pass through the point at infinity. Two straight lines intersect in general in *two* points, one of which is at infinity; while if the lines are parallel they must be regarded as tangent to each other at infinity. Etc.

Far less commonly recognized is what we get when we consider the geometry of inversion in the complex plane.* We must here consider the two sets of minimal lines

$$X + \sqrt{-1}\, Y = \text{const.}, \quad X - \sqrt{-1}\, Y = \text{const.}$$

It is readily seen that every motion of the plane in itself carries over a minimal line into a minimal line of the same set, while every reflection in a line† and every inversion in a circle‡ carries over a minimal line into a minimal line of the other set. In the case of inversions, however, there are two exceptions: points on the minimal lines through the center of the circle of inversion are not transformed at all, but may,

* Cf. Study: "Das Apollonische Problem," *Math. Annalen*, vol. 49 (1897), p. 497, and E. v. Weber: "Zur Theorie der Kreisverwandtschaften in der Ebene," *Bayerische Sitzungsberichte*, vol. 31 (1901), p. 367, in neither of which long treatments is a word said concerning the nature of the infinite region.

† Reflection is defined when and only when the line is not a minimal line.

‡ Inversion in a circle is defined when and only when the circle is not a null circle.

as in previous cases, be said to be thrown to infinity. Thus we are led to expect that in the geometry of inversion of the complex plane the infinite region should be taken as consisting of one real ideal point (the *point at infinity*, of the real plane) and through it two ideal minimal lines containing no other real point.*

All this can be put on a firm basis by the use of circular coordinates

$$\Xi = X + \sqrt{-1}\, Y, \quad \mathrm{H} = X - \sqrt{-1}\, Y.$$

In terms of these coordinates the group of direct circular transformations, that is, those which do not interchange the two sets of minimal lines, may be written†

$$\Xi' = \frac{\alpha_1 \Xi + \beta_1}{\gamma_1 \Xi + \delta_1} \qquad \begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{vmatrix} \neq 0,$$

$$\mathrm{H}' = \frac{\alpha_2 \mathrm{H} + \beta_2}{\gamma_2 \mathrm{H} + \delta_2} \qquad \begin{vmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{vmatrix} \neq 0.$$

We may also introduce homogeneous circular coordinates $(\xi_1 : \xi_2, \eta_1 : \eta_2)$ by means of the relations

$$\Xi = \frac{\xi_1}{\xi_2}, \qquad \mathrm{H} = \frac{\eta_1}{\eta_2},$$

when the 6-parameter group of direct circular transformations becomes

$$\rho \xi_1' = \alpha_1 \xi_1 + \beta_1 \xi_2, \quad \sigma \eta_1' = \alpha_2 \eta_1 + \beta_2 \eta_2,$$

$$\rho \xi_2' = \gamma_1 \xi_1 + \delta_1 \xi_2, \quad \sigma \eta_2' = \gamma_2 \eta_1 + \delta_2 \eta_2.$$

The indirect circular transformations come by combining these direct transformations with an interchange of the ξ's with the η's.

It will be seen that the formulæ just written are identical with those used in § 2, except that Ξ and H now take the place of X and Y, and that, for real points, Ξ and H are complex. Hence, disregarding questions of reality, the

* Cf. my book, Über die Reihenentwicklungen der Potentialtheorie, Leipzig, Teubner, 1894, p. 27, footnote. The whole treatment there given was written under the inspiration of Klein.

† Cf. E. v. Weber, loc. cit., p. 382.

general theorem at the beginning of this section follows again, this time without reference to quadric surfaces.

A systematic presentation of the geometry of inversion from the point of view here explained, namely that the infinite region consists of two minimal lines intersecting in a real point, would be most desirable. In it, two distinct circles intersect in only two points, the circular points at infinity of projective geometry having now no existence. Every circle, to be sure, has two ideal points, namely the imaginary points where it cuts the ideal minimal lines; but these points are in general different for different circles. In the case of concentric circles, which have no finite points in common, these ideal points are the same for one circle as for the other. Two distinct circles always intersect in two and only two points, which may coincide, in which case we say the circles are tangent; and the same is true of a circle and a line (not minimal), or of two lines (not minimal); but a minimal line meets a circle or an ordinary line in only one point, so that there is no possibility of contact here.

As a final illustration of the desirability of this point of view I mention the subject of foci. According to the ordinary definition, a focus of a plane curve is a point of intersection of two minimal lines, one of each set, both of which are tangent to the curve. The scope of this definition is not, however, coextensive in the projective plane and in the plane of the geometry of inversion, since contact *at infinity* means something very different in the two cases. Thus, in the projective plane, the center of a circle is its focus, since the minimal lines through the center touch the circle at the circular points at infinity. In the plane of the geometry of inversion, a circle has no focus (any more than a straight line has) since, as we have just noted, a minimal line cannot touch it. Since in the plane of the geometry of inversion the 6-parameter group of circular transformations is *without exception* a contact transformation, and carries minimal lines into minimal lines, it is clear that the relation of a focus to a plane curve is *always* invariant with regard to circular transformations.

The reader should contrast the perfect simplicity and generality of such statements as we have just been making with the far less perfect form the geometry of inversion takes when we use the projective plane and therefore have the

circular points at infinity as fundamental points of the transformations of the group.* In order to get the complete benefit of our point of view, a suitable terminology (which should be neither uncouth nor very extensive) should be introduced. In particular, some name should be given to the class of curves consisting of all circles, and all lines which are not minimal lines. If, following the analogy of § 2, we call such curves *bilinear curves* and those which consist of a pair of minimal lines *singular* bilinear curves, we may say: any two distinct non-singular bilinear curves intersect in two points either distinct or coincident. Let the reader enunciate *completely*, and *correctly* the facts in the projective plane which correspond to this, and he will get some idea of the gain in simplicity to which we have referred.†

§ 5. *Three-Dimensional Space.*

In one-dimensional geometry (the straight line) there is no distinction between the three kinds of spaces with which we have been dealing, the groups of transformations in the three cases being identical.

In two dimensions, we have seen that two of the three geometries differ only in questions of reality, while the third, projective geometry, is essentially different from them.

When we get above two dimensions, all three geometries are essentially different even without the introduction of questions of reality. For the sake of simplicity, we confine ourselves to three dimensions. Unless the contrary is stated, we suppose all quantities to be complex.

The three dimensional projective space with its plane at infinity is so familiar that a mere reference is sufficient. The group is, of course, the 15-parameter group of all non-singular collineations.

In the space of analysis we have the 9-parameter group

$$X' = \frac{\alpha_1 X + \beta_1}{\gamma_1 X + \delta_1}, \qquad Y' = \frac{\alpha_2 Y + \beta_2}{\gamma_2 Y + \delta_2}, \qquad Z' = \frac{\alpha_3 Z + \beta_3}{\gamma_3 Z + \delta_3},$$

* Cf. the classical treatment given from this point of view by Darboux in his book Sur une Classe remarquable de Courbes et de Surfaces algébriques, 1872.

† We mention, in passing, that it is perfectly possible to study projective geometry in the plane of analysis or in the plane of the geometry of inversion. The collineations will then, in general, be quadratic transformations with the double point at infinity as a fundamental point and the ideal ground-lines (or minimal lines) as fundamental lines.

where we suppose the determinant of each of these three linear functions to be different from zero. The planes

$$X = \text{const.}, \quad Y = \text{const.}, \quad Z = \text{const.}$$

we speak of as the three sets of ground-planes; and the lines parallel to the coordinate axes, as the three sets of ground-lines. We then introduce our ideal points, precisely as in §2, so that they completely fill three ideal ground-planes; one belonging to each set. These ground-planes intersect in three ideal ground-lines, one of each set; and all three ideal ground-planes (and also all three of the ideal ground-lines just mentioned) have one and only one point in common, the triple point at infinity. Our nine-parameter group is then, without exception, a one-to-one continuous point transformation. It carries over every ground-line (or ground-plane) into another ground-line (or ground-plane) of the same set. The whole subject may be developed farther along precisely the lines of §2.

Finally in the geometry of inversion we deal with the 10-parameter group of circular transformations generated by all motions, reflections in planes (not minimal), and inversions in spheres (not null). By entirely elementary computation we find that these generating transformations all carry over minimal lines into minimal lines, except that an inversion throws all the minimal lines through the center of the sphere of inversion to infinity. Moreover the points on these minimal lines are the only points which are not transformed. The minimal lines through a point form a minimal cone, and every minimal cone is carried into another minimal cone, with the exception just noted of the minimal cone with vertex at the center of inversion, which is thrown to infinity, and also with the exception of the minimal cones whose vertices are on the minimal cone just mentioned. Such a minimal cone becomes a minimal plane, and, conversely, every minimal plane becomes a minimal cone of the kind last mentioned. The generating transformations other than inversions of course carry over minimal planes into minimal planes.

Out of this complexity we can bring the most beautiful order by introducing ideal points of which one is real (the *point at infinity* of the real geometry of inversion) and all the others imaginary, and which fill completely a minimal cone with the real ideal point as vertex. A minimal cone being defined as the locus of all minimal lines through a given point,

the minimal planes now appear merely as special cases of minimal cones, namely those whose vertices are imaginary ideal points. We may now say, without exception, that *in the space of the geometry of inversion,* as thus formed, *the 10-parameter group of circular transformations is a one-to-one continuous transformation which carries over every minimal line into a minimal line.* All this may be put on a firm foundation most conveniently by using as an analytic instrument Darboux's *pentaspherical coordinates.*

The simplest class of surfaces in this geometry (we will call them linear surfaces) consists of all planes and spheres. Among these we distinguish the singular surfaces, namely the null spheres (or minimal cones) of which, as we have seen, the minimal planes form a special case. Not only will linear surfaces go into linear surfaces (remaining singular if they were so originally) but any two non-singular (and also any two singular) linear surfaces are equivalent with regard to the 10-parameter group of circular transformations.

Here, again, the *circle at infinity* of projective geometry has no existence. Every sphere has ideal points, namely the points of the curve in which it meets the ideal minimal cone, and this curve may properly be called a circle since it is the intersection of two spheres, namely the given sphere and any sphere concentric with it. Two spheres which are not concentric will, however, not pass through the same ideal circle, as they do in projective space, but through different ones.

We recalled at the beginning of §4 an important result of Klein. If we take as our quadric surface a real, non-singular quadric with real points but imaginary generators, or, in particular, a sphere, that result may, as is well known, be stated more precisely as follows:

When we consider questions of reality the geometry of inversion in the plane is abstractly identical with the projective geometry of a sphere with real non-vanishing radius and real center.

The import of this statement is not merely that there is a one-to-one correspondence between the points of the sphere and the points of the plane of the geometry of inversion of such a sort that real points correspond to real points, but also that there is a one-to-one correspondence between the circular transformations of the plane and the collineations of the spherical surface into itself. The most elementary way

of establishing this correspondence is, of course, by means of a stereographic projection of the plane on the sphere. Now this stereographic projection may also be regarded as an inversion of space which carries over the plane into the sphere. We thus look at the sphere as lying not in projective space but in the space of the geometry of inversion, and we are thus led to inquire whether the transformations of the sphere into itself with which we are concerned may not also be obtainable by means of a subgroup of the 10-parameter group of circular transformations of space. A count of constants ($10 - 4 = 6$) suggests that there should be a 6-parameter group of circular transformations of space which carry over the sphere into itself. A closer examination shows that this is actually the case, but that instead of there being a one-to-one correspondence between the circular transformations of the plane and the transformations of this subgroup, the correspondence is one-to-two; either of the two circular transformations of space which give the same transformations of the sphere into itself being obtainable from the other by following, or preceding, it by an inversion in the given sphere. Thus we get the geometry of inversion of the plane either by considering the 6-parameter group of collineations of projective space which carry over a real non-singular sphere into itself or the 6-parameter group of circular transformations of the space of inversions which carry over a real non-singular sphere (or a real plane) into itself.

If what has here been said does not persuade all readers of the desirability of using, on occasion, other spaces than projective space, it is hoped that it may at least make clear the desirability of stating explicitly in all cases what space it is that is being used. Papers dealing with transformations which throw points to infinity are not in a satisfactory form unless it is made clear what infinite region is assumed.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
August, 1913.

SHORTER NOTICES.

Taschenbuch für Mathematiker und Physiker. Unter Mitwirkung zahlreicher Fachgenossen, herausgegeben von FELIX AUERBACH und RUDOLF ROTHE. 3. Jahrgang, 1913. Leipzig und Berlin, B. G. Teubner, 1913. x + 463 pp. 6 M.

THE general aims and arrangement of this very useful little encyclopedia were explained by Professor J. B. Shaw in his review of the first volume, 1909, published in this BULLETIN, volume 16 (1910), page 321. The present volume opens with a biographical sketch of Friedrich Kohlrausch, and the principal other new contributions, according to the preface, are devoted to the following subjects: Calendar and astronomy, theory of aggregates, group theory and the Galois theory of equations, the last theorem of Fermat, integral equations and their applications, multiply valued functions and uniformization, international commission on the teaching of mathematics, analytical mechanics, theory of quanta, geodesy, crystallography, and general chemistry.

Among the other new features there appears a short historical list of prominent deceased mathematicians. A similar list of prominent physicists is expected to appear in the next volume. To enable the publishers to sell the book at a lower price the present volume contains about 100 pages less than its immediate predecessor. Most of this reduction has been made in the part relating to mathematics. The present volume is about the same size as the first volume of the series and is sold at the same price, while the second volume costs one mark more. The index and the table of contents of the present volume include references to the more important subjects treated in the earlier volumes but omitted in this. The fourth volume (1915) is expected to appear at the end of 1914.

Small books dealing with such immense subjects are very useful, not only to the man who has very little time for study yet wishes to keep in touch with the most important facts and advances, but also to the student who may wish to get a clear notion of some central features before entering into details. The two mathematical subjects to which the largest

amount of space is devoted in the present volume are theory of aggregates (12 pages) and integral equations with applications (8 pages). The contributions dealing with these two subjects are due to G. Hessenberg and O. Toeplitz respectively. The article on Fermat's last theorem (6 pages) is due to A. Fleck. This article was suggested by H. A. Schwarz, and should be especially welcome to editors of journals in view of the fact that many papers on this subject are now being offered for publication by authors who are not familiar with the literature.

While the number of references is not relatively as large as one would expect in a larger encyclopedia, yet this number is considerable, and sufficient to exhibit to the general reader the fact that rapid advances are being made, since many of the references are to the works of men who are still living. Such small books are extremely useful to secure a more rapid dissemination of important new knowledge and thus to secure for scientific workers a more general recognition. It seems unfortunate that we do not have a similar work in English. Possibly the success of this series may encourage some English publisher to fill the gap which is as serious with us as it was in Germany before the appearance of the first volume of this Taschenbuch.

G. A. MILLER.

Introduction a la Théorie des Nombres algébriques. By J. SOMMER, translated from the German by A. LEVY, with a Preface by J. HADAMARD. Paris, A. Hermann et Fils, 1911. x + 376 pp.

SOMMER'S "Vorlesungen über Zahlentheorie," which was published in 1907, was the first book to present in form really available to the beginner, the elements of the theory of algebraic numbers as developed by Kummer, Dedekind, and Hilbert. By devoting the greater part of his book to quadratic and cubic number realms, with applications to other branches of mathematics, Sommer rendered a real service to students who wish to become acquainted with this beautiful theory. The German edition has already been reviewed by Professor Ling,* whose favorable opinion has been borne out by the appearance of the French translation within four years of the publication of the original.

* Ling, BULLETIN, vol. 13, No. 3, December 1907.

The translation has been faithfully made and will be most welcome to students who find German the more difficult language. The words "revue et augmentée" which appear on the title page are not to be too seriously taken, for they are certainly not warranted by the changes that have been made. The introductory chapter, which is devoted to divisibility of integers, the φ-function, congruences, and Fermat's theorem, has been cut down from seventeen to fifteen pages. Concerning the demonstration of the unique factorization theorem for rational integers the translator says in a foot-note: "Cette démonstration étant enseignée dans toutes les écoles françaises, je me suis permis d'abréger ici un peu le texte de M. Sommer." In a few other places we find omissions of unimportant statements. In the list of examples given in section 16 for the determination of the class number, the translator has substituted for the realm $k(\sqrt{31})$, the realm $k(\sqrt{79})$, which has the same class number. The increase in the number of pages from 361 to 376 is due mainly to two things: In the French edition the page is shorter, and greater display is given to the equations, matters which add materially to the appearance of the book.

On page 188, at which point Sommer's proof that the special Fermat's equation

$$x^3 + y^3 = z^3$$

has no solution in integers begins, the translator has very generously changed the heading from "Entwicklung von Legendre" to "Démonstration de M. Sommer." The complete omission of the name of Legendre, who used the principle employed, as Sommer tells us, "in seiner einfachsten Form," seems scarcely warranted.

In a gracefully written preface M. Hadamard deplores the neglect—"si étrange qu'il puisse paraître dans la patrie d'Hermite"—into which the theory of numbers has fallen in France and expresses the hope that through the recently renewed activities in this direction at the University of Paris, and the fact that French students have now an adequate text at their disposal, conditions will soon change.

Unfortunately the book, which is printed in excellent type, on good paper, and with wide margins, is seriously marred by the fact that the proof reading has been abominably done. No less than forty-seven errors, to say nothing of omission of punctuation marks, have been found in the first thirty pages

of Chapter II. Of course, most of the errors would be easily detected, even by the casual reader, but when the text is made to speak of the discriminant of the realm as "le plus grand diviseur des nombres entiers du corps" (page 26), the beginner may have some trouble in supplying the omission. The same thing is true of the omission of the word "premier" in the statement of the theorem on page 65. In many places, as on page 49, where several equations are printed in one line without commas between, the right member of one and the left member of the following one appear as a product. Again, on page 65 where examples are given to illustrate the use of the symbol $\left(\frac{d}{p}\right)$ to determine in what way the principal ideal (p) can be broken up into ideal factors, we find the word "Corps" in a line by itself followed in the next line by

$$k(\sqrt{-5}) \quad m = -5 \quad d = -20$$

without any punctuation whatever, just as though m and d were necessary to define the realm.

The book would have been greatly improved for the general reader by printing the theorems in italics instead of in Roman characters.

The reader who glances over the table of contents and finds the entry "Index" will wonder if Frenchmen are reforming in the matter of indexes. But his surprise will be quickly turned to disappointment when he finds that the word is only a translation of Sommer's "Literatur-Verzeichnis" referring to the list of tables relating to the theory of numbers.

E. B. SKINNER.

An Introduction to the Infinitesimal Calculus—Notes for the use of Science and Engineering Students. By H. S. CARSLAW, Professor of Mathematics in the University of Sidney. Second edition, 1912. Longmans, Green and Co. xvi + 137 pp.

As indicated in the subtitle and in the preface, this little book is intended for first year students in the engineering schools of universities and technical colleges. It presumes a preparatory knowledge of trigonometry and elementary algebra, only. The first edition (1905, x + 103 pages) demanded a knowledge of infinite series for the deduction of the formulas for differentiating e^x and $\log x$; in the present edition

new methods are used in the text and the old proofs are placed in an appendix. Other changes are the addition of a section on repeated differentiation, one on fluid pressure, and a number of new exercises.

In Chapter I (16 pages) the rectangular coordinate geometry of the straight line is presented; in Chapter II (14 pages) the meaning of differentiation; in Chapter III (14 pages) the differentiation of algebraic functions and some general theorems; in Chapter IV (16 pages) the differentiation of the trigonometric and inverse trigonometric functions; in Chapter V (20 pages) the differentiation of the logarithmic and exponential functions, maxima and minima, partial derivatives; Chapter VI (10 pages), the conic sections; Chapter VII (15 pages), integration; Chapter VIII (22 pages), the definite integral and its applications.

This book is an example of a brief text which has not sacrificed accuracy of definition of the terms used, and should be, for the most part, legible to a first year student without the constant assistance of a preceptor; this will hardly be the case however when he encounters the term "rectangular hyperbola" (page 18), or "parabola" (page 19), or when he is expected to remember (page 45) that

$$\lim_{\theta=0} (\sin \theta)/\theta = 1.$$

First year students are not too young to be emphatically impressed with the notion that whenever an indicated operation seems to demand dividing by zero, no such demand is really made, but a special investigation is required. Thus, after proving (pages 4–5) that for every prescribed real number m, the equation $y = mx$ represents a straight line through the origin, it would seem better to omit the statement, "if $m = \infty$, the line is the axis of y," and state the converse: Every line through the origin, *except the axis of y*, can be represented by an equation of this form. Similar remarks apply to the solution (page 10)

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b}$$

of the equations

$$ax + by + c = 0, \quad a'x + b'y + c' = 0,$$

and to the formula (page 37)

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

The notion of a limit set forth on page 21 is carefully worded and sufficient for the purpose immediately in view, and the use of the notation

$$\underset{x \to a}{Lt}(y) = b \quad \text{instead of} \quad \underset{x=a}{Lt}(y) = b$$

makes for clearness with a beginning student, but it would be unfortunate to leave him with the impression that a function never attains its limit. Continuity is not mentioned except incidentally in § 13, page 19. Little emphasis is put on the existence of a limit; in finding the derivative of $y = x^{p/q}$ (page 39) and of $y = \sin^{-1} x$ (page 52) the question of existence, which would be unlikely to arise in the mind of the reader, is not raised by the author.

Considerable interest attaches to the use in Chapter V of 7, 8, and 15 place logarithm tables to make it seem plausible to the reader (the author states in advance that the discussion is not a rigorous proof) that $\lim_{n=\infty} (1 + 1/n)^n$ exists, that $e = 2.718$ (approximately) and that $\lim_{n=\infty} n \log_{10} (1 + 1/n)^n$ exists and $= .4343$ (approximately). From these results

$$\frac{d}{dx} \log_{10} x = \frac{\log_{10} e}{x} \quad \text{and} \quad \frac{d}{dx} \log_e x = \frac{1}{x}$$

are deduced; then $de^x/dx = e^x$, considering e^x as the inverse of $\log_e x$.

If one decides not to go into a detailed proof to establish these formulas, it is of course a matter of opinion what would better be assumed and what proved. An alternative and equally plausible assumption, after e^x has been defined, is that

$$1 + x < e^x < \frac{1}{1 - x} \quad \text{for } -1 < x < 1,$$

from which follow

$$\frac{d}{dx} e^x = e^x \quad \text{and} \quad \left(\frac{d}{dx} \log_e x \right)_{x>0} = \frac{1}{x},$$

considering $\log_e x$ as the inverse of e^x.

A. M. KENYON.

Monographs on Topics of Modern Mathematics Relevant to the Elementary Field. Edited by J. W. A. YOUNG. New York, Longmans, Green and Co., 1911. viii + 416 pp.

THIS book contains nine monographs by as many authors, as follows:

1. The Foundations of Geometry, by Oswald Veblen. Pages 1–51.
2. Modern Pure Geometry, by Thomas F. Holgate. Pages 53–89.
3. Non-Euclidean Geometry, by Frederick S. Woods. Pages 91–147.
4. The Fundamental Propositions of Algebra, by Edward V. Huntington. Pages 149–207.
5. The Algebraic Equation, by G. A. Miller. Pages 209–260.
6. The Function Concept and the Fundamental Notions of the Calculus, by Gilbert Ames Bliss. Pages 261–304.
7. The Theory of Numbers, by J. W. A. Young. Pages 305–349.
8. Constructions with Ruler and Compasses; Regular Polygons, by L. E. Dickson. Pages 351–386.
9. The History and Transcendence of π, by David Eugene Smith. Pages 387–416.

The authors of these monographs were of the opinion that there is room for a serious effort to bring within the reach of secondary teachers, college students, and others of a like stage of mathematical advancement, a scientific treatment of some of the regions of advanced mathematics that have points of contact with the elementary field. They felt that a great need of our secondary instruction in mathematics is the enlargement of the mathematical horizon of the teachers themselves; and they believed that there is a large body of earnest teachers and students that are eager to extend their mathematical knowledge if the path can be made plain and feasible for them.

The object of these monographs is to make a contribution toward meeting this need. That the topics are well selected for this purpose may be seen from the foregoing list. The aim of each monograph is to bring the reader into touch with some characteristic results and viewpoints of the topic considered, and several of them point out the bearing of these matters on elementary mathematics.

The material in each of the monographs falls more or less definitely into three parts. First, there is a considerable body of results proved in full. Next, there is a statement without proof of some of the further leading methods and results, so as to give in minimum compass a bird's-eye view of the whole subject. Finally, in connection with most of the monographs there is a small number of references indicating what the reader may profitably take up after he has mastered the contents of the monograph itself.

Naturally, the amount of technical mathematical knowledge presupposed on the part of the reader varies with different subjects. For the reading of a large part of the book, a knowledge of elementary algebra and geometry, together with a certain measure of mathematical maturity, is sufficient. Since the various papers are written by men well qualified to speak on the several subjects, there is much in them that will repay careful and detailed study by advanced students.

There is not space here to go into an analysis of the separate monographs of the book. It seems desirable, however, to indicate a few specific criticisms.

The symbol $\{ABC\}$ is used in two senses in the first monograph (cf. assumption II and theorem 7); it would seem better to avoid this. The proof of theorem 19 in the same paper is incomplete.

The statement on page 265 that the word function was originally used to denote any power of a number seems to be inaccurate. Compare Encyclopédie, tome II, volume 1, page 3.

In the seventh monograph we have the definition, "A prime number (or briefly, a prime) is a number having no other factors than itself and unity." According to this definition, unity is a prime number. But on page 311 we find $\varphi(1) = 1$, $\varphi(p) = p - 1$, where p is a prime—an obvious inconsistency if 1 is a prime. To the reviewer it seems better to exclude unity from the list of primes.

The proof reading on some of the monographs was not carefully done; see, for example, the inconsistent use of italics on page 11. But the errors of this sort will usually cause the reader no serious inconvenience, and consequently no list of them is supplied here.

In conclusion I should like to say with emphasis that this book makes a step in the right direction. It looks forward to the time when the secondary teacher will know some of the

most modern notions concerning fundamental mathematical disciplines, a precursor of that day when the undergraduate curriculum will contain, in their more elementary aspects, many of those subjects and ideas which make mathematics a thing of esthetic delight to those who are now laboring in its development.

Another valuable contribution to the same end would be a treatise on elementary geometry written from the point of view of the first monograph of the present book. How this may well be done can be seen from the nature and arrangement of this monograph.

R. D. CARMICHAEL.

Higher Algebra. By H. E. HAWKES. Boston, Ginn and Company, 1913. v + 222 pp.

THE subjects treated in a course in algebra designed for freshmen and advanced secondary students constitute almost a fixed unit; as to the manner of presenting these subjects there is some difference of opinion. Some teachers believe in carefully formulating a few assumptions and building upon these with absolute rigor. From the standpoint of the scientist this is possibly the only correct view. Some have asserted that this thoroughly rigorous method of proving every step is practical as well as theoretically elegant; but by far the greater number of teachers have found by experience that an entirely different method of procedure is preferable. The average freshman does not have the intensive interest of the scientist in the subject; he is looking for general rules rather than the exceptions with which the scientist is vitally concerned; the interest of the student should be awakened and stimulated by frequent appeals to his intuition and by giving the subject a real and tangible basis; any long series of purely logical steps should be avoided if possible; hence, it has been found desirable in presenting the subject to this type of student to make bold and explicit assumptions as they become necessary in the development, and to postpone proofs of a severely logical character to a later and more critical study.

Professor Hawkes has written his book consistently from the second of the viewpoints just described. The book has been prepared to meet the needs of the student who will continue his mathematics as far as the calculus. The author has adapted the book both to the engineer and to the student

of pure mathematics by using the topics which must be emphasized by the engineer, such as numerical computations, checks, graphical methods, the use of tables, and the solution of specific problems, as a basis on which the foundations of the more theoretical portions of algebra might be suitably and advantageously laid.

In the introductory review the usual topics are taken up, but in such a way that much may be brought out which has not permeated the mind of the student in passing over the work for the first time. Considerable attention is paid to the substitution of values of x in polynomials by a selection of a number of good exercises in this work. To factor an expression of the type $ax^2 + bx + c$ the author separates b into two parts whose product is ac; i. e., $6x^2 - 13x - 5 \equiv 6x^2 - 15x + 2x - 5 \equiv 3x(2x - 5) + (2x - 5) \equiv (2x - 5)(3x + 1)$. This is an improvement upon the method of multiplying through by the coefficient of x^2 and changing the variable, after which it must be changed back to x.

Probably the most salient feature of the book is the fact that the author insists upon the student's acquiring a knowledge of the fundamental theory of the equation; this aim appears not only in the chapter on the theory of equations, where it is woven into the very texture of the material, but also in the treatment of other topics. Some intimation of the theory of the quadratic is given on the first page of the book, where attention is called to the fact that the factors of $x^2 + bx + c$ are $(x + p)(x + q)$, where p and q are two numbers whose sum is b and whose product is c. Later, to find the condition that one root of a quadratic shall be n times the other is given as an exercise; the condition that the two roots of the quadratic shall be equal follows as a special case, but the discriminant is later given the prominence that it deserves. Some emphasis is laid upon the parameters which occur in quadratics: application is made of this in finding algebraically the maximum and minimum of a quadratic function; i. e., in the expression $y = ax^2 + bx + c$, the y is transposed and considered a parameter, and that value of y is obtained which makes the discriminant of the resulting expression equal to zero.

The notion of function, the graph of a function and especially the slope of the graph are clearly explained and attractively illustrated. Much use is made of graphical

methods. An excellent example of the use of graphs is given on page 58 in discussing the behavior of a quadratic function when the coefficient of x^2 approaches zero as a limit. It is easier to remember the method of expanding a three-rowed determinant by re-writing its first two columns, taking the right hand diagonals for positive terms and the left hand diagonals for negative terms, than the usual but more complicated method of forming these diagonals given by the author. The changing of x to $y - 2$ in separating into partial fractions expressions of the type $(3x^2 - 4x + 3)/(x + 2)^3$ is of so much advantage that one wonders why so many books fail to suggest it. That the fraction a/n approaches a limit when a is a constant and n a variable which becomes infinite is brought out in a dialogue between two speakers and serves as a relief from the hackneyed expressions which usually occur in that connection.

I believe that the author may be fairly criticized for not having given a more formal discussion of undetermined coefficients. If one is trying to find topics which must be used by the applied scientist and which may be used as a medium in which the foundations of theoretical algebra might be laid—the expressed intention of the author—I know of no subject which could be better used to advantage in this connection than undetermined coefficients.

The book is exceptionally free from typographical errors. I have noticed only one; on page 21, in the example at the bottom of the page, 3 75 should be 3.75.

All of the subjects taken up in this book except probability and infinite series have been treated in the author's former book on Advanced Algebra. Criticism of these topics over a period of about seven years has resulted in much improvement. It is my belief that the teacher will find the Higher Algebra a good text to follow closely in courses designed to give the student a thoroughly good workable knowledge of this portion of algebra.

JOSEPH EUGENE ROWE.

The Teaching of High School Mathematics. By GEORGE W. EVANS. Boston, Houghton Mifflin Company, 1911. x+94 pp. Price 35 cents.

THE little book under review is one of the Riverside Educational Monographs. When the reader meets in the pref-

ace a discussion of the modern demands of "the ideal of practicality," "sound educational sociology," and "sound rational psychology," he is led to expect some quite radical suggestions as to the material and methods to be used in high school mathematics. However one finds in the modern demands upon the teacher of mathematics, as discussed by the author, nothing very radical and only what progressive teachers have been practicing for some years; the large number of teachers who still believe the chief value of mathematics to be disciplinary, and who cannot accept all the claims made against the doctrine of formal discipline, will accept the suggestions of the author as helpful to better mathematics teaching.

The emphasis placed upon efficiency and self-reliance in computation, the equation as the central idea in algebra, geometry as a source of algebraic material, the importance of graphical work, broader foundations of proof in geometry, and a saner attitude toward the method of limits in elementary mathematics are now generally endorsed by progressive teachers. Possibly the most radical suggestions are those for the use of Simpson's rule for plane areas and the principle of Cavalieri.

The book gives the final impression of being written by a successful and progressive teacher; the few hours required to read it will give a teacher some valuable suggestions as well as inspiration to improve his own teaching.

ERNEST B. LYTLE.

Koordinatensysteme. Von PAUL B. FISCHER. Leipzig, Sammlung Göschen, 1911. 125 pp,

THIS little book gives a very readable and, on the whole, satisfactory account of the most important systems of coordinates which have been used in geometry. The discussion of cartesian point coordinates, Plückerian line and plane coordinates leads the author naturally to the homogeneous coordinates of Hesse and to the general projective systems of coordinates. He then discusses some of the most elementary properties of curvilinear coordinates in general and gives some more detailed account of certain special systems, especially polar and elliptic coordinates.

Although Fischer formulates the general notion of coordinates, one cannot help remarking how little has, as yet, been accomplished in the direction of a general theory of

coordinates. Consequently, a book on coordinate systems, like the present one, necessarily lacks unity since a fundamental unifying principle, such as the group concept in the theory of transformations, has not yet been found.

E. J. WILCZYNSKI.

NOTES.

THE official list of officers and members of the American Mathematical Society will be published in January. Blanks for furnishing necessary information were sent out some time ago. To insure accuracy, members are requested to inform the Secretary at once of any changes of status or address.

THE following changes in the editorial staff of the *Transactions* will soon take place: Professor H. S. WHITE retires from the Editorial Board on February 1, and will be succeeded by Professor P. F. SMITH. Professor W. R. LONGLEY and Dr. R. L. MOORE have been appointed associate editors. Professor ARTHUR RANUM has consented to serve as associate editor until Professor HUTCHINSON is able to resume his work.

AT the annual meeting of the London mathematical society held on November 13, the following officers were elected for the present academic year: president, A. E. H. LOVE; vice-presidents, H. F. BAKER and W. BURNSIDE; secretaries, J. H. GRACE and T. J. BROMWICH. Also two members of the council. The following papers were presented at the meeting. By G. T. BENNETT, "The skew-isogram mechanism"; by G. H. HARDY and J. E. LITTLEWOOD, "Tauberian theorems concerning power series the coefficients of which are positive"; by G. H. HARDY, "Lambert's theorem"; by J. E. CAMPBELL, "The connection between surfaces the lines of curvature of which are spherical and surfaces the inflectional tangents of which belong to linear complexes," and "Surfaces the systems of inflectional tangents of which belong to systems of linear complexes"; by W. H. YOUNG, "Integration with respect to a function of bounded variation"; by W. W. JOHNSON, "The computation of Cotes's numbers, and their values up to $n = 20$"; by S. G. SOAL, "Some ruler constructions for the covariants of a binary quantic"; by T. C. LEWIS, "Analogues of orthocentric tetrahedra in higher space."

THE third Scandinavian congress of mathematicians was held at Christiania September 3–6 under the presidency of Professor C. STÖRMER. The following papers were presented at the congress: N. NIELSEN, "Some applications of Bernoulli's numbers to the theory of numbers"; A. WIMAN, "The relation between the maximum modulus and the largest term of an analytic function"; G. STRÖMBERG, "Analysis of the temperature curve for Stockholm during the period 1894–1911"; G. MITTAG-LEFFLER, "A theorem of Abel and Dirichlet's series"; J. HJELMSLEV, "Geometry of reality"; K. F. SUNDMAN, "Apparatus for finding special perturbations"; J. F. STEFFENSEN, "Analytic representation of sums in the theory of numbers"; N. E. NÖRLUND, "Facultative series"; L. VEGARD, "Action of a gravitational field of force upon certain solutions"; P. HEEGAARD, "Contributions to the theory of graphs"; M. RIESZ, "On Fourier's series"; E. PHRAGMEN, "Remarks on the preceding congress"; W. V. JENSEN, "A formula in the theory of numbers," and "On the roots of algebraic equations"; C. JUEL, "Toroidal non-algebraic surfaces of the fourth order"; V. BJERKNES, "Mathematical treatment of meteorological problems"; J. HJELMSLEV, "Geometric experience"; M. T. SKOLEM, "Constitution of groups of the identical calculus"; A. PALMSTRÖM, "Calculation of invalid insurance"; S. JOHANSSON, "Representation of automorphic potentials"; HOLTSMARK, "The calculation of insurance on two lives."

THE Swiss mathematical society held its fourth annual meeting at Frauenfeld, September 9, under the presidency of Professor H. FEHR. The following papers were presented at the meeting. By L. CRELIER, "On correspondences in synthetic geometry"; by R. FUETER, "On algebraic equations with given groups"; by G. DUMAS, "On the singularities of surfaces"; by A. SPEISER, "On the factorization of algebraic forms"; by L. BIEBERBACH, "A new method of conformal representation"; by E. MARCHAND, "Newton's rule in the theory of algebraic equations"; by F. RUDIO, "Report on the publication of Euler's works"; by D. MIRIMANOFF, "Some points in the theory of sets"; by W. H. YOUNG, "The integral of Stieltjes and its generalization"; by A. EINSTEIN, "Physical basis and directing ideas of a theory of gravitation"; by M. GROSSMANN, "Mathematical

definitions, methods, and problems connected with the theory of gravitation."

THE Société physico-mathématique of Kasan offers a Lobachevsky prize of 500 roubles for the most important work on non-euclidean geometry published during the six years prior to November 4, 1915.

IN Teubner's abbreviated catalogue, issued in October, 1913, appear, among others, the following announcements. The second part of volume one of the second German edition of Pascal's Repertorium der höheren Mathematik is to appear in the spring of 1914; the second part of the second volume is promised before New Year's. In the Grundlehren der Mathematik für Studierende und Lehrer two further volumes are being prepared: Algebra, by E. Netto, and Geometrische Gebilde vom Standpunkte der Verwandtschaften, by W. F. Meyer. The investigations of the Bolyai (W. and J.), by P. Stäckel, is in the press; Geheimnisse der Rechenkünstler by P. Maenchen will appear in December. Twenty-seven volumes on advanced mathematics are either in the press or in advanced preparation. Of the encyclopedia, eleven parts of the German edition and twenty-five of the French are in the press.

THE twenty-first meeting of the Association of teachers of mathematics in the middle states and Maryland was held at Albany, November 29, 1913, extending through a morning and afternoon session. The following papers were presented, "Are particular abilities necessary for the pupil to gain an understanding of the elementary and secondary mathematics as usually given at the present time?," by M. J. BABB and C. F. WHEELOCK; "A comparison at equal school ages of the attainments in mathematics of the European and American schoolboy with a consideration of causes and remedies," by J. C. BROWN; "Mathematics as a means to culture and discipline," by A. D. YOCUM; "The use of the question in the classroom," by R. STEVENS.

At the business meeting, Mr. E. R. SMITH, of the Park School, Baltimore, was elected president, Professor H. E. HAWKES of Columbia University, vice-president, and Mr. H. F. HART, of the Montclair high school, secretary.

The association comprises five sections, whose meeting places are at Philadelphia, Pittsburg, New York, Rochester, and Syracuse, respectively.

THE firm of Martin Schilling in Leipzig announces sixteen new models of curves of constant width (series XL, numbers 4 to 19) by Professor F. Schilling (price M 145); a plaster model of the surface of centers of the cubic asymptotic surface $xyz = a^3$ (series XLIII, number 8), by Professor E. Beutel (price M 120); five approximations curve from the theory of Fourier series (series L, numbers 1–5), by Mr. H. Küstner and Dr. L. Föppl (price, M 7.50).

THE announcement, in the November BULLETIN, of the mathematical courses given at the technical school of Stuttgart during the present semester should have included the following: Professor R. MEHMKE: Point analysis, with exercises, four hours.

THE royal society of Edinburgh has elected as honorary fellows Professors H. LAMB of the University of Manchester and V. VOLTERRA, of the University of Rome.

PROFESSOR F. KLEIN, of the University of Göttingen, has been elected to membership in the academies of Berlin, Bucharest, and Naples.

PROFESSOR D. HILBERT, of the University of Göttingen, has been elected corresponding member of the royal academy of sciences at Berlin.

PROFESSOR K. BOEHM, of the University of Heidelberg, has been appointed professor of mathematics at the University of Königsberg, as successor to Professor G. FABER, who has accepted a similar position at the University of Strassburg.

AT the University of Agram Professor G. MAJCEN has been promoted to a full professorship of mathematics.

DR. R. WEITZENBÖCK, of the University of Vienna, has been appointed docent in mathematics at the University and the Technical School of Gratz.

DR. K. BERLINER and DR. L. HENOCH have been appointed docents in mathematics at the University of Bern.

DR. A. PROELL, of the technical school at Danzig, has been appointed professor of mechanics at the German technical school at Brünn.

HON. BERTRAND RUSSELL, lecturer at Trinity College, Cambridge University, has been appointed lecturer in philosophy at Harvard University for the current year. Beginning early in March, he will give courses in the theory of knowledge and advanced logic.

AT the College of the City of New York Dr. F. G. REYNOLDS has been promoted from an assistant professorship to an associate professorship of mathematics. Dr. M. PHILIP has been promoted to an assistant professorship, and Mr. G. M. BRETT to an instructorship.

PROFESSOR J. F. DOWNEY, of the University of Minnesota, will retire at the end of the present academic year.

AT the University of Michigan assistant professors PETER FIELD and T. R. RUNNING have been promoted to junior professorships and Dr. C. E. LOVE to an assistant professorship of mathematics. Dr. TOMLINSON FORT, Mr. C. H. FORSYTHE, Mr. BARNEM LIBBY, and Mr. L. J. ROUSE have been appointed instructors in mathematics.

AT the University of Illinois Professor H. L. RIETZ has been promoted to a full professorship of mathematics; Mr. H. C. ZEIS has been appointed assistant in mathematics. Mr. G. E. CARSCALLEN has resigned to accept the professorship of mathematics at Hiram College.

MR. C. W. WESTER has been appointed instructor in mathematics at the Univerity of Iowa.

MR. C. P. MILLS has been appointed instructor in mathematics in the Carnegie Institute of Technology.

MISS FLORENCE KENDALL and Mr. W. E. EDINGTON have been appointed instructors in mathematics at the University of Colorado.

PROFESSOR F. POCKELS, of the University of Heidelberg, died August 29, at the age of 60 years.

DR. H. VALENTINER, formerly of the University of Copenhagen, died September 17, at the age of 62 years.

SIR ROBERT STAWELL BALL, Lowndean professor of astronomy and geometry in Cambridge University and for the last twenty-one years director of the Cambridge observatory, died November 25, at the age of 73 years.

DR. FRANZ KOLÁČEK, professor of mathematical physics in the Bohemian University of Prague, died December 8, at the age of 62 years.

PROFESSOR JOHN EASTMAN CLARKE, of Boston University, died November 22, at the age of 63 years. Dr. Clarke had been a member of the American Mathematical Society since 1900.

DR. H. J. MESSENGER, actuary of the Travelers Insurance Company, died December 15, at the age of 58 years. Dr. Messenger had been a member of the American Mathematical Society since 1889.

CATALOGUES of second-hand mathematical books.—Gustav Fock, Schlossgasse 7–9, Leipzig, catalogue 448, about 4700 titles in mathematics.—W. Heffer and Sons, Cambridge, England, catalogue 104, about 3500 titles in mathematics and science.—Henry Sotheran & Co., 140 Strand, London, catalogue 741, about 1500 titles in mathematics and science.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADHÉMAR (R. D'). Leçons sur les principes de l'analyse. Avec une note de S. Bernstein. Tome 2: Fonctions synectiques. Méthode des majorantes. Equations aux dérivées partielles du premier ordre. Fonctions elliptiques. Fonctions entières. Paris, Gauthier-Villars, 1913. 8vo. 7+298 pp. Fr. 10.00

ARCAIS (F. D'). Analisi infinitesimale. 3a edizione, con modificazioni ed aggiunte. Volume II (ultimo). Padova, Draghi, 1913. 8vo. 8+941 pp.

ARCHIMEDES. Opera omnia. Cum commentariis Eutocii iterum edidit I. L. Heiberg. Band II. Leipzig, Teubner, 1913. 8vo. 18+554 pp. M. 8.00

——. Werke. Mit modernen Bezeichnungen herausgegeben und mit einer Einleitung versehen von T. L. Heath. Deutsch von F. Kliem. Berlin, Haering, 1913. 8vo. 12+477 pp. M. 16.00

BRAGDON (C.). A primer of higher space (the fourth dimension). Rochester, N. Y., Manas Press, 1913. 8vo. Cloth. $1.00

BROWER (L. E. J.). Intuitionisme et formalisme. (Discours sur l'intuitionisme et le formalisme dans la philosophie mathématique.) Amsterdam, Clausen, 1913. 8vo. 32 pp.

CAUCHY (A.). Oeuvres complètes d'Augustin Cauchy, publieés sous la direction scientifique de l'Académie des sciences. 2e series, tome 11. Paris, Gauthier-Villars, 1913. 4to. 512 pp. Fr. 25.00

CHATELET (A.). Leçons sur la théorie des nombres: Modules, entiers algébriques, réduction continuelle. Paris, Gauthier-Villars, 1913. 8vo. 10+156 pp. Fr. 5.50

COULON (R.). Etude sur la géométrie des formes naturelles. Rouen, Lainé, 1913. 8vo. 74 pp.

ELLIOTT (E. B.). An introduction to algebra of quantics. 2d edition, London, Clarendon Press, 1913. 8vo. 432 pp. Cloth. 15 s.

EULER (L.). Opera omnia. Series I, Vol. XI: Institutiones calculi integralis. Ediderunt F. Engel et L. Schlesinger. Vol. 1. Leipzig. Teubner, 1913. 8vo. 18+462 pp. M. 28.00

——. Opera omnia. Series I, Vol. X: Institutiones calculi differentialis. Edidit G. Kowalewski. Leipzig, Teubner, 1913. 4to. 676 pp. M. 32.00

EVANS (G. C.). Sopra l'algebra delle funzioni permutabili. Roma, 1912. 4to. L. 2.50

FIRMICUS MATERNUS (J.). Matheseos libri VIII. Ediderunt W. Kroll, F. Skutsch, et K. Ziegler. Fasciculus 2: Libri 4 posteriores cum praefatione et indicibus. Leipzig, 1913. 8vo. 70+558 pp. M. 12.00

FRISCHEISEN-KÖHLER (M.). Wissenschaft und Wirklichkeit. (Wissenschaft und Hypothese. Nr. 15.) Leipzig, Teubner, 1912. 8vo. 8+478 pp. M. 8.00

FUETER (R.). Die Klassenkörper der komplexen Multiplikation und ihr Einfluss auf die Entwicklung der Zahlentheorie. Leipzig, Teubner, 1911. 8vo. 47 pp. M. 1.50

GIGLI (D.). Questioni riguardanti le matematiche elementari. Dei numeri complessi a due e a più unità. Bologna, Zanichelli, 1912. 8vo. 146 pp.

HEATH (T. L.). See ARCHIMEDES.

HEIBERG (I. L.). See ARCHIMEDES.

KREMPELHUBER (F. v.). Eine neue Mathematik und Naturphilosophie. Braunschweig, 1913. 8vo. 3+152 pp. M. 5.00

LONGLEY (W. R.). Tables and formulas for solving numerical problems in analytical geometry, calculus and applied mathematics. Boston, Ginn, 1913. 12mo. 5+31 pp. $0.50

MARKUS (C.). Das Gesetz der metaphysischen Dimensionen. Ein Beweis für das Theorem des Fermat. 2te, vermehrte Auflage. Berlin, 1913. 8vo. 4 pp. M. 1.50

PAPERS from the science laboratories of the University of Sydney, 1908–9 to 1911–12. A: From the departments of mathematics, physics, chemistry and engineering. Sydney, Australia, University of Sydney, 1912.

STAUDE (O.). Analytische Geometrie der kubischen Kegelschnitte. (Teubner's Sammlung. Band XXXVIII.) Leipzig, Teubner, 1913. 8vo. 8+242 pp. Cloth. M. 10.00

STENTZEL (A.). Jesus Christus und sein Stern. Eine chronologische Untersuchung. Hamburg, 1913. 8vo. 8+240 pp. M. 6.00

VOLTERRA (V.). Leçons sur les fonctions de lignes, professées à la Sorbonne en 1912. Recueillies et rédigées par J. Péres. (Collection de monographies sur la théorie des fonctions publiée sous la direction de E. Borel.) Paris, Gauthier-Villars, 1913. 8vo. 6+230 pp. Fr. 7.50

WEITZEL (C. G.). Unterrichtsbriefe zur Einführung in die höhere Mathematik, 9te–16te Lieferung. Wien, Hartleben, 1913. M. 4.00

WILSON (E. B.). Vector analysis. Textbook for the use of students of mathematics and physics. Founded upon the lectures of J. W. Gibbs. New Haven, Conn., Yale University, 1913. 8vo. 18+436 pp. Cloth. $4.00

II. ELEMENTARY MATHEMATICS.

ABBOTT (P.). Exercises in arithmetic and mensuration. London, Longmans, 1913. 8vo. 534 pp. 3s. 6d.

ANDEREGG (F.) and ROE (E. D. JR.). Trigonometry for schools and colleges. Revised edition by F. Anderegg. Boston, Ginn, 1913. 12mo. 9+108 pp. $0.75

BAKER (W. M.) and BOURNE (A. A.). A shorter algebra. With answers. London, Bell, 1913. 8vo. 388 pp. 2s. 6d.

BARKER (E. H.). Computing tables and mathematical formulas. Boston, Ginn, 1913. 16mo. 5+88 pp. $0.75

BARNARD (S.) and CHILD (J. M.). Key to a new algebra. Volume 2, containing parts 4, 5 and 6. London, Macmillan, 1913. 8vo. 6s. 6d.

BAUER (G. W.) and BROOKE (W. E.). Plane and spherical trigonometry. Boston, Heath. 8vo. 12+164 pp. Half leather. $1.50

BEKE (E.) und MIKOLA (A.). Abhandlungen über die Reform des mathematischen Unterrichts in Ungarn. Leipzig, Teubner, 1911. 8vo. 6+160 pp. M. 4.00

BOREL (E.). Die Elemente der Mathematik. Deutsch von P. Stäckel. 2tes Heft: Aufgaben aus der Geometrie. Lösungen. Leipzig, Teubner, 1913. 8vo. 4+39 pp. M. 1.50

BOURNE (A. A.). See BAKER (W. M.).

BRANFORD (B.). Betrachtungen über mathematische Erziehung vom Kindergarten bis zur Universität. Deutsche Bearbeitung von R. Schimmack und H. Weinreich. Leipzig, Teubner, 1913. 8vo. 8+403 pp, Cloth. M. 7.00

BRIGGS (W.) and BRYAN (G. H.). The tutorial algebra or radhakrishnan. 4th edition. 8th impression. London, Clive, 1913. 8vo. 8+647 pp. 6s. 6d.

BROOKE (W. E.). See BAUER (G. W.).

BRYAN (G. H.). See BRIGGS (W.).

BUSH (W. N.) and CLARKE (J. B.). The elements of plane geometry. New York, Silver, Burdett & Co., 1913. 12mo. 12+239 pp. $0.75

CALFEE (J. E.). Rural arithmetic. A course in arithmetic intended to start children to thinking and figuring on home and its improvement. Boston, Ginn, 1913. 12mo. 7+119 pp. $0.30

CHILD (J. M.). See BARNARD (S.).

CLARKE (J. B.). See BUSH (W. N.).

DRESSLER (H.). Mathematische Lehrmittelsammlungen, insbesondere für höhere Schulen. (Unterrichtskommission. Berichte, Nr. 9.). Leipzig, Teubner, 1913. 8vo. Pp. 187–217. M. 1.00

GENAU (A.). Mathematische Ueberraschungen für Lehrer und Rechenfreunde. Arnsberg, Stahl, 1913. 8vo. 80 pp. M. 1.00

HART (W. W.). See WELLS (W.).

JELÍNEK (L.). Mathematische Tafeln. 6te Auflage. Wien, Pichler, 1913. 8vo. 174 pp. Cloth. M. 2.20

KATZ (D.). Psychologie und mathematischer Unterricht. (Abhandlungen über den mathematischen Unterricht in Deutschland. III 8.) Leipzig, Teubner, 1913. 8vo. 4+120 pp. M. 3.20

MIKOLA (A.). See BEKE (E.).

NUNN (P.). Exercises in algebra, including trigonometry. Part I. London, Longmans, 1913. 8vo. 434 pp. 4s.

ROE (E. D. JR.). See ANDEREGG (F.).

WELLS (W.) and HART (W. W.). Second course in algebra. Boston, Heath, 1913. 12mo. 5+285 pp. Cloth. $0.90

WERNICKE (A.). Mathematik und philosophische Propädeutik. (Unterrichtskommission. Abhandlungen. III 7.) Leipzig, Teubner, 1912. 8vo. 7+138 pp. M. 4.00

III. APPLIED MATHEMATICS.

ALBRECHT (M. F.) und VIEROW (C. S.). Lehrbuch der Navigation und ihrer mathematischen Hilfswissenschaften. 10te Auflage, bearbeitet von G. Holtz. Berlin, 1913. 8vo. M. 14.00

ARISTARCHUS of Samos (the ancient Copernicus). History of Greek astronomy to Aristarchus, together with Aristarchus's treatise on the sizes and distances of the sun and moon. New translation and notes by T. L. Heath. Oxford, 1913. 8vo. 432 pp. Cloth. 18 s.

BECKMANN (A.) und NIEBOUR (H.). Tafeln zur Ermittelung der Invaliden- und Altersrenten. Neue Ausgabe. Berlin, 1913. 8vo. 9+230 pp. M. 6.00

BJÖRNBO (A.). See WERNER (J.).

BLANCHARD (A. H.) and DROWNE (H. B.). Text-book on highway engineering. New York, Wiley, 1913. 8vo. 13+762 pp. Cloth. $4.50

BORTKIEVICZ (L. v.). Die radioaktive Strahlung als Gegenstand wahrscheinlichkeits-theoretischer Untersuchungen. Berlin, Springer, 1913. 8vo. 7+84 pp. M. 4.00

BURALI-FORTI (C.) e MARCOLONGO (R.). Analyse vectorielle générale. Volume II: Applications à la mécanique et à la physique. Pavia, Mattei, 1913. 8vo. 12+143 pp. L. 5.00

BUTAVAND (F.). Les lois empiriques du système solaire et les harmoniques tourbillonnaires. Paris, Gauthier-Villars, 1913. 8vo. 50 pp. Fr. 1.75

CARR (W. K.). Matter and some of its dimensions. New York, Harper, 1913. 16mo. 120 pp. Cloth. $0.75

CARUS (P.). The principle of relativity in the light of the philosophy of science. Chicago, Open Court, 1913. 8vo. 105 pp. $1.00

CHEVASSUS (R.). See PLANCK (M.).

CHWOLSON (O. D.). Lehrbuch der Physik. 4ter Band: Die Lehre von der Elektrizität. 2te Hälfte, 1te Abteilung. Braunschweig, Vieweg, 1913. 8vo. 450 pp. M. 7.50

COHEN (L.). Formulae and tables for the calculation of alternating current problems. New York, McGraw-Hill, 1913. 8vo. 282 pp. Cloth. $3.00

DELAUNAY. Etudes astronomiques: Lois sur les distances des satellites; force répulsive des astres; etc. Paris, 1912. 8vo. 64 pp. Fr. 2.00

DELAUNAY (A.). Les bases théoriques de l'aviation. Paris, Imhaus et Chapelot, 1913. 8vo. 46 pp. Fr. 1.50

DROWNE (H. B.). See BLANCHARD (A. H.).

EHRENFEST (P.). Zur Krise der Lichtäther-Hypothese. (Antritts-Rede.) Berlin, Springer, 1913. 8vo. 23 pp. M. 0.60

EINSTEIN (A.). See LORENTZ (H. A.).

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome VII, 1re volume: Astronomie sphérique. Fascicule 1: E. Anding-H. Bourget, Système de référence et mesure du temps.—A. Bemporad-P. Puiseux, Réfraction et extinction.—F. Cohn-E. Doublet-L. Picard, Réduction des observations astronomiques.—C. W. Wirtz-G. Fayet, Détermination de la longitude et de la latitude. Leipzig, Teubner, 1913. 8vo. Pp. 1–224. M. 8.40

HALL (G. H.). Elementary theory of alternate current working. (*The Electrician.*) London, 1913. 8vo. 202 pp. 3s. 6d.

HEATH (R. S.). A text-book of elementary statics. London, Clarendon Press, 1913. 8vo. 288 pp. 4s. 6d.

HEATH (T. L.). See ARISTARCHUS.

JÖRGENSEN (N. R.). Grundzüge einer Theorie der Lebensversicherung. Jena, 1913. 8vo. 10+408 pp. M. 12.00

LÁSKA (W.). Astronomie und die mathematische Geographie. 2ter Teil: Praktische und theoretische Astronomie. Bremerhaven, Vangerow, 1913. 8vo. 8+164 pp. M. 5.00

LE DANTEC (L. M.). Théorie géométrique et mécanique de l'hélice-turbine. Paris, Dupont, 1913. 4to. 77 pp. Fr. 4.00

LORENTZ (H. A.), EINSTEIN (A.) und MINKOWSKI (H.). Das Relativitätsprinzip. Eine Sammlung von Abhandlungen. Mit Anmerkungen von A. Sommerfeld und Vorwort von O. Blumenthal. (Fortschritte der mathematischen Wissenschaften in Monographien, 2tes Heft.) Leipzig, Teubner, 1913. 8vo. 4+89 pp. Cloth. M. 3.60

MANITIUS (C.). See PTOLEMÄUS.

MARCHAND-BEY (E. E.). Mécanique générale. Erreurs et lacunes à la base de la mécanique classique actuelle. Principes et formules nouveaux complémentaires. (*Mécanique-Electricité.*) Paris, Loubat, 1913. 8vo. 8+103 pp.

MARCOLOGNO (R.). See BURALI-FORTI (C.).

MELLOR (J. W.). Higher mathematics for students of chemistry and physics, with special reference to practical work. 4th edition, enlarged. New York, Longmans, 1913. 8vo. 21+641 pp. Cloth. $4.50

MINKOWSKI (H.). See LORENTZ (H. A.).

MOORE (C. J.). Logarithmic reduction tables. For students of analytical chemistry. Boston, Ginn, 1913. 12mo. 8+78 pp. Cloth. $1.00

MÜLLER (O.). Graphisches Rechnen und die graphische Darstellung. Glauchau, Streit, 1913. 8vo. 64 pp. M. 2.00

NIEBOUR (H.). See BECKMANN (A.).

PASQUIER (L. G. DU). Mathematische Theorie der Invaliditätsversicherung. Bern, 1913. 8vo. 17+107 pp. M. 2.50

PLANCK (M.). Leçons de thermodynamique. Traduit sur la 3e édition allemande (augmentée) par R. Chevassus. Paris, Hermann, 1913. 8vo. 315 pp.

PRESCOTT (J.). Mechanics of particles and rigid bodies. London, Longmans, 1913. 8vo. 544 pp. 12s. 6d.

PROCLUS (DIADOCHUS LYCIUS). Institutio physica. Edidit et interpretatione germanica commentarioque instruxit A. Ritzenfeld. Leipzig, Teubner, 1912. 8vo. 16+78 pp. M. 2.20

PTOLEMÄUS (CLAUDIUS). Handbuch der Astronomie. Aus dem Griechischen übersetzt und mit erklärenden Anmerkungen versehen von C. Manitius. Band II. Leipzig, Teubner, 1913. 8vo. 6+446 pp. M. 8.60

REED (S. J.). Turbines applied to marine propulsion. New York, Van Nostrand, 1913. 4to. 8+174 pp. Cloth. $5.00

REYNAUD (P.). La rotation des astres déduite de l'attraction. Paris, 1912. 8vo. 36 pp. Fr. 2.00

RIPPER (W.). Heat engines. New edition. New York, Longmans, 1913. 8vo. 10+352 pp. Cloth. $1.10

RITZENFELD (A.). See PROCLUS.

SCHNELL (H.). Aufgaben zur graphischen Darstellung. Braunschweig, 1913. 8vo. 8+43 pp. Cloth. M. 1.60

STILL (A.). Overhead electric power transmission. New York, McGraw-Hill, 1913. 8vo. 10+310 pp. Cloth. $3.00

THOMPSON (P.). Navigation; a method of finding a ship's position at sea by one observation only. New York, Longmans, 1913. 8vo. 56 pp. Cloth. $1.20

TRAUTWINE (J. C.). The civil engineer's pocket-book. Revised by J. C. Trautwine, jr. and J. C. Trautwine, 3d. 19th edition. Philadelphia, Trautwine, 1913. 12mo. 1257 pp. $5.00

VIEROW (C. S.). See ALBRECHT (M. F.).

WERNER (J.). Ioannis Verneri de triangulis sphaericis libri quatuor, de meteoroscopiis libri sex, cum proemio G. I. Rhetici. II: De meteoroscopiis. Herausgegeben von J. Würschmidt unter Benutzung der Vorarbeiten von A. Björnbo. Mit einem Vorwort von E. Wiedemann. (Abhandlungen zur Geschichte der mathematischen Wissenschaften. Heft 24 II.) Leipzig, Teubner, 1913. 8vo. 5+260 pp. M. 12.00

WÜRSCHMIDT (J.). See WERNER (J.).